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Brief Notes and Comments

Problem Department

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THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer.

Papers intended for a particular Department should be sent to the Chairman of that Department.

An Important Diagnosis

The contrast between the increasing importance of mathematics in science and technology and the decreasing interest in mathematics on the part of the average youth at the secondary level is a condition which would appear to be worthy of careful diagnosis.

Is the phenomenon due to the widespread practice in the high school of student options in course materials? Is it due to a widely prevailing assumption on the part of school administrations that only a few minds can acquire much mathematics beyond the elementary levels? Is it the consequence of a world-wide utilitarianism that values only the applications of a mathematical principle while it scorns the disciplines of thought that *discover* the mathematical principle? Is it the result of a steadily growing repugnance of the youthful mind to the concentrations ordinarily demanded by mathematical study? Is it due to an excessive vocationalizing of the curricula of the pre-collegiate schools? The probability that most, if not all, of these things are jointly the cause of the chasm between the mathematics-using technician and the mathematics-shy youth, constitutes a challenge to the mathematical fraternity to assume the role of leadership in all post-war educational movements.

S. T. Sanders.

The Mathematics of a Nut Cutter

By Joseph B. Reynolds
Lehigh University

There is a machine for cutting nuts in the shapes of regular polygons which is remarkable for its simplicity in principle and accuracy in results. It consists essentially of two parts that rotate in the same sense about parallel fixed axes. One pa t carries the cutting edge and the other the work or stock from which the nut is to be cut. The motions of these parts are simply related; the former turns at twice the speed of the latter and the cutting edge moves, at the point of contact, in a direction nearly opposite to that of the same point on the work.*

The notable feature of the process is the accuracy with which the cut resulting from the combined motions approaches a straight line. It is the purpose of this article to develop the mathematics of this machine and compute the small variat:ons from perfectly flat sides resulting on nuts of different numbers of sides.

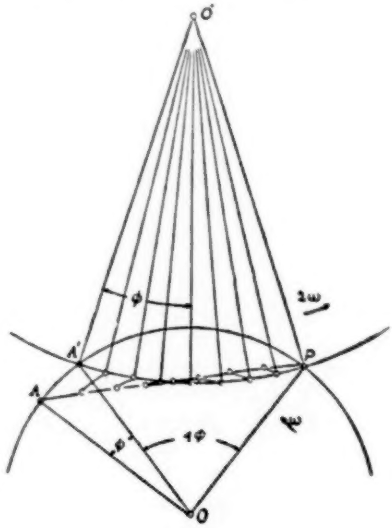


Fig. 1.

In Fig. 1 the part carrying the cutting tooth P rotates about the center O' while the round stock from which the nut is being cut

*The physical facts concerning this machine were furnished the author by W. W. Johnson of Cleveland, Ohio.

rotates about the center O. The dimensions are such that the angle $A'O'P$ is half the angle $A'OP$. When the cutting edge started to make the cut AP it was at A' which was then the position of A. Since the work turns at half the rate of the cutting part, the angle AOA' is half $A'O'P$ or one-fourth $A'OP$. Hence, if angle AOA' is $\varnothing$, $A'O'P$ is $2\varnothing$, $A'OP$ is $4\varnothing$ and AOP is $5\varnothing$. Between A' and P on Fig. 1 are shown eight successive positions of the cutting edge and on AP the eight corresponding positions on the work. It is the path AP, one side of the nut, that closely approximates a straight line.

It is seen that $5\varnothing$ measures the angle spanned by one side of the nut at the center of the stock and that there are half as many cutting teeth as sides on the nut. For a two-part machine this excludes odd-numbered sides for the nuts but we shall develop the theory for the cases of triangular, square, pentagonal, hexagonal and octagonal shapes. For these shapes the values of $\varnothing$ are, respectively, 24°, 18°, 14.4°, 12° and 9°.

In Fig. 2 is shown the cutting tooth P when it has rotated through an angle 2θ from first contact with the work, in which time the work has rotated about O through an angle θ. Taking O as origin and OA as X-axis we read from the figure the coordinates of the position of P. They are

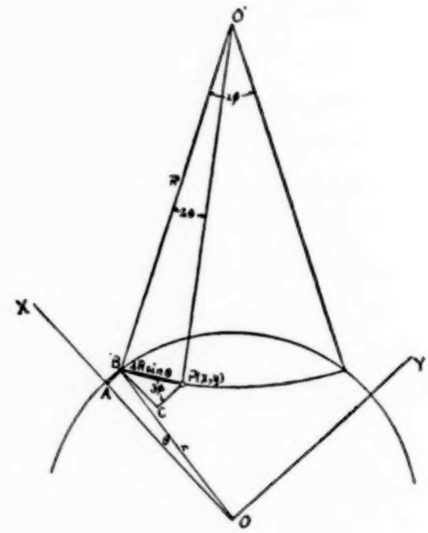


FIG. 2.

$$x = OA - BC = r\cos\theta - 2R\sin\theta\sin 3\varnothing$$

$$y = AB + CP = r\sin\theta + 2R\sin\theta\cos 3\varnothing$$

in which r is the radius of the round stock and R the radius of the path of the cutting tooth. Since $R/r = \sin 2\varnothing / \sin \varnothing = 2 \cos \varnothing$, these may be written

$$x + 4r \sin \theta \cos \varnothing \sin 3\varnothing = \cos \theta$$

$$y = r \sin \theta (1 + 4 \cos \varnothing \cos 3\varnothing)$$

If now we write k for $1 + 4 \cos \varnothing \cos 3\varnothing$ and $\tan A$ for $4 \cos \varnothing \sin 3\varnothing$ the xy-equation can be written in the form

$$k^2x^2 + 2kxy \tan A + y^2 \sec^2 A = k^2r^2.$$

This is an ellipse with center at the origin, major axis on the line $y + x \cot(5\varnothing/2) = 0$ and minor axis on the line $y = x \tan(5\varnothing/2)$.

The orientation of the ellipse with respect to the stock and cutter is shown in Fig. 3. The cut is that portion of the elliptic arc lying

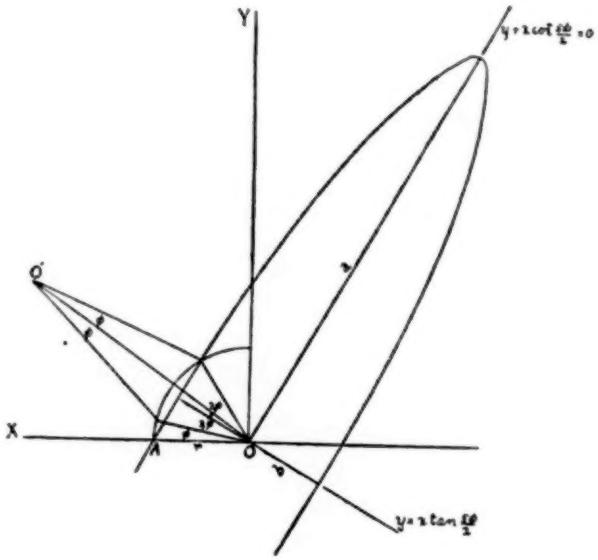


FIG. 3.

between the X-axis and the line $y = x \tan 5\varnothing$. This arc segment is symmetrical to the minor axis, so that the maximum variation of the cut from a straight line is

$$D = b - r \cos(5\varnothing/2)$$

in which $2b$ is the length of the minor axis of the ellipse. Letting $2a$ be the length of the major axis, we find the following relations between a, b and the radius r of the stock.

$$a/r = (1+4 \cos \varnothing \cos 3\varnothing)/[1-4 \sin \tfrac{1}{2}\varnothing \sin(3\varnothing/2)]$$

and
$$b/r = (1+4 \cos \varnothing \cos 3\varnothing)/[1+4 \cos \tfrac{1}{2}\varnothing \cos(3\varnothing/2)].$$

In the table below are listed values of a/r and b/r for five shapes of nuts. The value of D, the greatest variation from a straight line, is also given. The values tabulated are for the case when the nuts are cut from stock 2 inches in diameter.

Shape	a/r	b/r	*D in inches* ($r=1$)
Triangle	4.1653	.5112	.0112
Square	4.5202	.7159	.0088
Pentagon	4.6898	.8155	.0065
Hexagon	4.7833	.8708	.0048
Octagon	4.8774	.9267	.0028

Fig. 4 shows the position of the related ellipse with respect to each shape of nut.

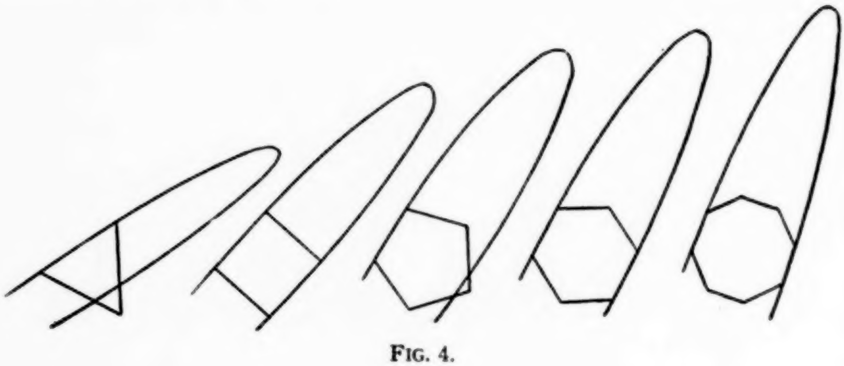


FIG. 4.

Taylor's Formula and Sterling's Numbers

By Tomlinson Fort
University of Georgia

1 Taylor's series is as follows

$$f(x+h)=f(x)+f'(x)\frac{h}{1!}+f''(x)\frac{h^2}{2!}+\cdots \tag{1}$$

If $\Delta f(x)=f(x+h)-f(x)$ and

$$Df(x)=\frac{d}{dx}f(x)$$

we can write Taylor's series thus

$$\Delta f(x)=\frac{h}{1!}Df(x)+\frac{h^2}{2!}D^2f(x)+\cdots\ . \tag{2}$$

In other words Taylor's series serves to express the operator Δ in terms of the operator D. Now let us represent $\Delta(\Delta f(x))$ by $\Delta^2 f(x)$ and so on. The problem of expressing Δ^k in terms of D immediately presents itself. The following formula is easy to prove by mathematical induction as well as by other methods.

$$\Delta^k f(x)=\sum_{i=0}^{k}(-1)^i{}_kC_{k-i}f(x+(k-i)h) \tag{3}$$

where ${}_kC_{k-i}$ is a binomial coefficient

Now $$f(x+2h)=f(x)+\frac{2h}{1!}Df(x)+\frac{(2h)^2}{2!}D^2f(x)+\cdots$$

and similarly for $f(x+3h)$, $\cdots$. We can substitute these series in (3) and collect coefficients. Consequently, in a formal way what we have undertaken to do is surely possible. We wish to determine the resulting coefficients.

In order to avoid all questions of convergence we let $f(x)$ be a polynomial of degree m. Then

$$f(x+t)=f(x)+\frac{t}{1!}Df(x)+\frac{t^2}{2!}D^2f(x)+\cdots$$
$$+\frac{t^m}{m!}D^mf(x).$$

Consider t as a variable and take the kth difference retaining h as difference interval.

$$(4) \qquad \Delta^k f(x+t) = \frac{\Delta^k t^k}{k!} D^k f(x) + \frac{\Delta^k t^{k+1}}{(k+1)!} D^{k+1} f(x) + \cdots$$
$$+ \frac{\Delta^k t^m}{m!} D^m f(x).$$

In (4) let $t=0$.

$$(5) \qquad \Delta^k f(x) = \frac{\Delta^k O^k}{k!} D^k f(x) + \frac{\Delta^k O^{k+1}}{(k+1)!} D^{k+1} f(x) + \cdots$$
$$+ \frac{\Delta^k O^m}{m!} D^m f(x).$$

This is the desired formula. If we let

$$\mathfrak{S}_j^k = \frac{1}{k!} \Delta^k O^j, \quad h=1$$

and note that $\Delta^k O^j]_{h=h} = \Delta^k O^j]_{h=1} \cdot h^j$, we can write (5) as follows:

$$(6) \qquad \Delta^k f(x) = h^k \mathfrak{S}_k^k D^k f(x) + \frac{h^{k+1}}{k+1} \mathfrak{S}^k_{k+1} D^{k+1} f(x) + \cdots$$
$$+ \frac{h^m}{(k+1)\cdots m} \mathfrak{S}^k_m D^m f(x).$$

The numbers $\mathfrak{S}_j^k$ are known as Sterling's numbers of the second kind. If in (3) we replace $f(x)$ by x^j and let $x=0$ and $h=1$ we have

$$\mathfrak{S}_j^k = \frac{1}{k!} \sum_{i=0}^{k-1} {}_kC_{k-i}(k-i)^j = \frac{1}{k!} \sum_{i=1}^{k} {}_kC_i i^j.$$

The converse problem, namely to express D^k in terms of Δ, now presents itself. If we notice that $\mathfrak{S}_k^k = 1$ and set up equation (6) for $k=1,\cdots, m$, we have a set of equations in $Df(x),\cdots, D^m f(x)$ where the determinant of the coefficients is 1. Consequently solution is always possible. However, an easier procedure is as follows. We develop $f(x+t)$ by Newton's formula. We have

$$(7) \qquad f(x+t) = f(x) + \frac{1}{h} \frac{t^{(1)}}{1!} \Delta f(x) + \frac{1}{h^2} \frac{t^{(2)}}{2!} \Delta^2 f(x) + \cdots$$
$$+ \frac{1}{h^m} \frac{t^{(m)}}{m!} \Delta^m f(x)$$

where $t^{(j)} = t(t-h)\cdots(t-(j-1)h)$. Newton's formula is an analogue of Taylor's formula and is found in any book on interpolation. Differentiate k times with respect to t.

$$(8)\quad D^k f(x+t) = \frac{1}{h^k}\,\frac{D^k t^{(k)}}{k!}\Delta^k f(x) + \frac{1}{h^{k+1}}\,\frac{D^k t^{(k+1)}}{(k+1)}\Delta^{k+1} f(x) + \cdots$$

$$+\frac{1}{h^m}\,\frac{D^k t^{(m)}}{m!}\Delta^m f(x).$$

Let $t=0$.

$$(9)\qquad D^k f(x) = \frac{1}{h^k}\,\frac{D^k O^{(k)}}{k!}\Delta^k f(x) + \frac{1}{h^{k+1}}\,\frac{D^k O^{(k+1)}}{(k+1)!}\Delta^{k+1} f(x) + \cdots$$

$$+\frac{1}{h^m}\,\frac{D^k O^{(m)}}{m!}\Delta^m f(x).$$

This is the desired formula. If we let

$$S_j^k = h^{k-j}\frac{D^k O^{(j)}}{k!},$$

we can write

$$(10)\qquad D^k f(x) = \frac{1}{h^k}\left(S_k^k \Delta^k f(x) + \frac{1}{k+1} S^k{}_{k+1}\Delta^{k+1} f(x) + \cdots \right.$$

$$\left. + \frac{1}{(k+1)\cdots m} S_m{}^k \Delta^m f(x) \right).$$

The numbers S_j^k are called Sterling's numbers of the first kind. It is immediate that S_i^k equals the coefficient of x^k in the expansion of $x(x-1,\cdots(x-(j-1))$.

We now wish to extend formulas (4) and (8) to functions that are not polynomials, developing formulas for the remainder analogous to the well-known forms in Taylor's formula. We suppose $f(x)$ a function having a derivative of order $(m+1)$ at all points considered. Begin with a consideration of (8). Let

$$(11)\qquad \varphi(t) \equiv f(x+t) - f(x) - \frac{1}{h}\,\frac{t^{(1)}}{1!}\Delta f(x) - \cdots$$

$$-\frac{1}{h^m}\,\frac{t^{(m)}}{m!}\Delta^m f(x) - Kt^{(m+1)}$$

where K is a constant as yet unspecified. Now $\varphi(t)=0$ when $t=0, h, \cdots, mh$. The last term vanishes at each of these points due to the presence of $t^{(m+1)}$. The rest of the expression vanishes in as

much as Newton's formula is exact at these points. Choose a positive integer $k<m+1$. By successive application of Rolle's Theorem we see that $D^k\varphi(t)$ vanishes at at least $m+1-k$ points on the interval $0<t<mh$. Now let T be a point distinct from any of these such that $D^kT^{(m+1)}\neq 0$. Then determine K so that $D^k\varphi(T)=0$. In order to be assured that T be distinct from at least $m+1-k$ zeros of $D^k\varphi(t)$, on the interval $0<t<mh$, we assume that it does not belong to this interval although it may be an end point. With this assumption $D^k\varphi(t)$ has at least $m+2-k$ zeros on the interval delimited by $T, 0,\cdots, mh$. Hence, by repeated application of Rolle's Theorem, $D^{m+1}\varphi(t)$ vanishes at least once on the interval delimited by $T, 0,\cdots, mh$. Hence

$$K=\frac{1}{(m+1)!}D^{m+1}f(x+\xi).$$

In particular we can choose $T=0$ since $D^kO^{(m+1)}\neq 0$, $k<m+1$. This is surely the case as D^kO^{m+1} is simply the coefficient of t^{m+1-k} in the expansion of $t^{(m+1)}=t(t-h)\cdots(t-mh)$. We have the formula

$$D^kf(x+T)=\frac{1}{h^k}\frac{D^kT^{(k)}}{k!}\Delta^kf(x)+\cdots \tag{12}$$

$$+\frac{1}{h^m}\frac{D^kT^{(m)}}{m!}\Delta f(x)+R_m{}^k,\quad k<m+1.$$

$$R_m{}^k=D^{m+1}f(x+\xi)\frac{D^kT^{(m+1)}}{(m+1)!}, \tag{13}$$

where, as we have stated, T does not lie within the interval $0<t<mh$ and is so chosen that $D^kT^{(m+1)}\neq 0$, where moreover ξ is within the interval delimited by $T, 0,\cdots, mh$. In particular if $T=0$ we have

$$D^kf(x)=\frac{1}{h^k}\left[S_k{}^k\Delta^kf(x)+\frac{1}{k+1}S^k{}_{k+1}\Delta^{k+1}f(x)+\cdots\right. \tag{14}$$

$$\left.+\frac{1}{(k+1)\cdots m}S^k{}_m\Delta^mf(x)\right]+R_m{}^k.$$

$$R_m{}^k=h^{m+1-k}\frac{S_{m+1}{}^k}{(h+1)\cdots m}D^{m+1}f(x+\xi),\quad T=0. \tag{15}$$

Now let us consider (4) and again suppose that $f(x)$ is a function such that $D^{m+1}f(x)$ exists at all points considered. Assume $k<m+1$. Form the function,

$$\psi(t) \equiv f(x+t) - f(x) - \frac{t^m}{1!}Df(x) - \frac{t^2}{2!}D^2f(x) - \cdots$$
$$- \frac{t^m}{m!}D^mf(x) - \overline{K}\frac{t^{m+1}}{(m+1)!}$$

where $\overline{K}$ is a constant as yet unspecified. This function has a zero of order $(m+1)$ when $t=0$. Consequently $D^k\psi(t)$ has a zero of order $(m+1-k)$ when $t=0$. Now let $\overline{T}$ be chosen such that $\Delta^k\overline{T}^{m+1} \neq 0$. Then choose $\overline{K}$ so that $\Delta^k\psi(t)$ vanishes when $t=\overline{T}$. But by the mean value theorem $\Delta^k\psi(\overline{T}) = h^kD^k\psi(\delta)$ where $\overline{T} < \delta < \overline{T}+kh$. Consequently $D^k\psi(\delta)=0$ where $\overline{T} < \delta < \overline{T}+kh$. We additionally restrict $\overline{T}$ so that this interval does not include the origin although the origin may be an end point. Under these restrictions $D^k\psi(t)$ vanishes $m+2-k$ times in the interval delimited by 0, $\overline{T}$, $\overline{T}+kh$. Hence $D^{m+1}\psi(t)$ vanishes at least once on this interval. Hence

$$\overline{K} = \frac{1}{(m+1)!}D^{m+1}f(x+\xi)$$

where ξ is on the interval delimited by 0, $\overline{T}$, $\overline{T}+kh$. It is well to remark that T can have any positive value. This is true since $\Delta^k\overline{T}^{m+1} = h^kD^k\eta^{m+1} \neq 0$ because $\eta > \overline{T} > 0$ and $k < m+1$.

We consequently have the formula

$$\Delta^kf(x+\overline{T}) = \frac{\Delta^k\overline{T}^k}{k!}D^kf(x) + \cdots + \frac{\Delta^k\overline{T}^m}{m!}D^mf(x) + \mathfrak{R}_m{}^k \tag{16}$$

$$\mathfrak{R}_m{}^k = D^{m+1}f(x+\xi)\frac{\Delta^h\overline{T}^{m+1}}{(m+1)!} \tag{17}$$

Where $\overline{T}$ is restricted as above delineated. If $\overline{T}=0$ we have

$$\Delta^kf(x) = h^k\mathfrak{S}_k{}^kD^kf(x) + h^{k+1}\frac{\mathfrak{S}_{k+1}{}^k}{k+1} + \cdots \tag{18}$$
$$+ h^m\frac{\mathfrak{S}_m{}^k}{(k+1)\cdots m}D^mf(x) + \mathfrak{R}_m{}^k$$

$$\mathfrak{R}_m{}^k = h^{m+1}\frac{\mathfrak{S}_{m+1}{}^k}{(k+1)\cdots(m+1)}D^{m+1}f(x+\xi). \tag{19}$$

Remainder terms as given in (15) and (19) are known as Markoff's forms.

Formulas for the remainder which have been obtained are analogous to the Lagrange form for the remainder in Taylor's formula. The problem of obtaining an interval (Cauchy) form for the remainder suggests itself. The following formula follows from (3) by the subtraction of

$$\sum_{i=0}^{k} (-1)^i \, {}_kC_{k-i} f(x) = (1-1)^k f(x) = 0$$

(20) $$\Delta^k f(x) = \sum_{i=0}^{k-1} (-1)^i \, {}_kC_{k-i} [f(x+(k-i)h - f(x))].$$

Now in (20) expand $f(x+(k-i)h)-f(x)$ by Taylor's formula using the integral form for the remainder. Denote the sum of these remainders by $R_m^{(k)}$. Then

(21) $$R_m^k = \sum_{i=0}^{k-1} (-1)^i \, {}_kC_{k-i} \frac{1}{m!} \int_x^{x+(k-i)h} (x+(k-i)h-t)^m D^{m+1} f(t) dt$$

$$= \frac{1}{m!} \Delta_z^k \int_z^z (z-t)^m D^{m+1} f(t) dt]_{z=x}.$$

Here Δ_z^k means the difference with z as variable. This formula is believed to be new. A form for the remainder in (16) somewhat similar to (21) is obtainable also from the Cauchy formula. It seems too complicated, however, to be of service.

2. We now shall start over again so to speak. Let us call attention to formula (2). This can be symbolically written

$$\Delta f(x) = (e^{hD}-1) f(x).$$

We write symbolically $\Delta = (e^{hD}-1)$. If the Taylor's series converges this symbolic interpretation can be given a genuine significance. If $f(x)$ is a polynomial there is no question as to the interpretation of the operator $(e^{hD}-1)$ nor as to the fact that its successive application obeys all the laws of ordinary multiplication. More specifically:

Assume $f(x)$ of degree m. All terms in the formal expansion of $(e^{hD}-1)$ of higher power in D than m will yield zero when operating on $f(x)$. We then in fact are dealing only with

$$hD + \frac{(hD)^2}{2!} + \cdots + \frac{(hD)^m}{m!},$$

an operator of a finite number of terms. We write

(22) $$\Delta^k f(x) = (e^{hD}-1)^k f(x).$$

Equation (22) is valid for the polynomial and is an abbreviated way to write (5). But if $k>0$,

$$(e^x-1)^k=\left(x+\frac{x^2}{2!}+\cdots+\frac{x^m}{m!}\right)^k+(\text{terms of degree higher than } m).$$

Since m is any positive integer, from (5)

$$(23)\qquad (e^x-1)^k=\mathbf{S}_k{}^kx^k+\frac{\mathbf{S}^k{}_{k+1}}{k+1}x^{k+1}+\frac{\mathbf{S}^k{}_{k+2}}{(k+1)(k+2)}x^{k+2}+\cdots$$

We now consider the inverse problem. We note that

$$S_j{}^1=(-1)^{j-1}(j-1)!$$

Consequently by (14)

$$Df(x)=\frac{1}{h}[\Delta f(x)-\tfrac{1}{2}\Delta^2f(x)+\tfrac{1}{3}\Delta^3f(x)-\cdots]$$

an equation surely valid if $f(x)$ is a polynomial. Symbolically we write

$$(24)\qquad Df(x)=\frac{1}{h}\log(1+\Delta)f(x)$$

or

$$D=\frac{1}{h}\log(1+\Delta).$$

Exactly as in the previous case

$$D^kf(x)=\frac{1}{h^k}(\log(1+\Delta))^kf(x).$$

This is a symbolic method of writing (10).

The function $1/k!(e^x-1)^k$ is called a generating function for Sterling's numbers of the second kind of order k. Its development into a power series affords a method of calculating these numbers. Similarly $1/k!(\log(1+x))^k$ is a generating function for Sterling's numbers of the first kind of order k.

A variety of relationships between the Sterling's numbers exist but it is not proposed to make any study of these here. An interested person can consult the books cited below. However, we do remark the following two relations

$$(24)\qquad \mathbf{S}^m{}_{j+1}=m\mathbf{S}_j{}^m+\mathbf{S}_j{}^{m-1}$$

$$(25)\qquad S^m{}_{j+1}=S_j{}^{m-1}-jS_j{}^m.$$

The reader may be interested in proving them for himself.

3. Let us look to generalization of the Sterling numbers and to the more interesting problem of expressing one linear operator in term

of another. In particular it is to be observed that we applied the operator Δ to Taylor's series. This is nothing unusual about Δ and operators other than Δ can be expressed in terms of D by the aid of Taylor's series alone. Similarly other operators than D can be expressed in terms of Δ by means of Newton's formula. We note next that Taylor's formula and Newton's formula are both very special cases of a formula obtained by the author and published in the *Bulletin of the American Mathematical Society*. This formula can be made to replace Taylor's and Newton's formulas of the text and by means of it a variety of operators can be expressed in terms of others by methods strictly analogous to those used in this paper. This formula is

$$Z(x+t)=\sum_{v=0}^{m}\frac{F_v(t)}{v!}P^vQP^{-k}Z(x)+R, \tag{26}$$ where

$$R=\frac{1}{(m+1)!}p(t)D^{m+1}Z(\xi(t)). \tag{27}$$

The functions and operators are more fully described in the *Bulletin*. This formula is Taylor's formula if $P=Q=D$ and $F_v(t)=t^v$ and Newton's formula if $P=Q=\Delta$ and $F_v(t)=t^{(v)}$. Now if we are given a linear operator V and operate with V on both sides of (26) as an equation in t and then let $t=0$ assuming that

$$VZ(x+t)\big]_{t=0}=VZ(x)$$ we have

$$VZ(x)=\sum_{v=0}^{m}\frac{VF_v(t)}{v!}\bigg]_{t=0}P^vQP^{-k}Z(x)+R^1, \tag{28}$$ where

$$R^1=\frac{1}{(m+1)!}V[p(t)D^{m+1}Z(\xi(t))]_{t=0}. \tag{29}$$

This formula serves to express V in terms of PQP^{-k}. If P and Q are such operators that $P=Q$ and $k=1$, we have expressed V in terms of P. The remainder when the operators are applied to $Z(x)$ is given by (29). Formula (29) is not so convenient as (13). Moreover, the methods employed in the derivation of (13) are much more widely applicable than to that formula alone. For example instead of the forward difference operator we could have used the divided difference, the backward difference, or the central difference operator. However, formula (29) is very general and gives additional light on (13) and (19).

BIBLIOGRAPHY

A Treatise on the Calculus of Finite Differences by George Boole—Chapter II.

Calculus of Finite Differences by Charles Jordan—Chapter IV.

Interpolation by J. F. Steffensen—Sections 6 and 7.

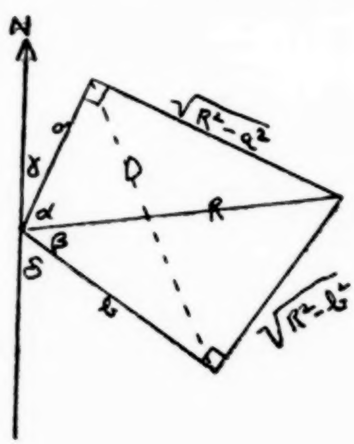
Generalizations of the Bernoulli Polynomials and Numbers and Corresponding Summation Formulas by T. Fort; *Bulletin* of the American Mathematical Society; August, 1942, also December, 1942.

Maximum Dips by Seismic Methods

By William Funkenbusch
Computer, Western Geophysical Company

In locating subsurface domes in our oil exploration work, it is necessary to determine the magnitude and direction of dip of a rock bed at a certain point. In the usual case this was found as the resultant vector of a rectangle. The two component vectors were determined in direction by the setting out of the receivers (jugs), and in magnitude by the time lag in the impulse at different jugs along the row. In the diagram the bearing of the jug lines a and b are given by γ and δ respectively. Since we are free, in most all cases, to choose γ and δ, we have $\gamma+\delta=90°$.

A case of interest from a mathematical viewpoint is however $\gamma+\delta\neq 90°$. In this case the line of maximum dip does not follow the Parallelogram Law of forces, but the strike lines are always at right angles to the jug lines. The problem then is given γ,δ,a, and b to determine α, R, and Θ, where Θ equals the angle of maximum dip so that $\Theta=\tan^{-1}R$. We will also say $\Phi=\tan^{-1}\alpha$ and $\psi=\tan^{-1}b$.



Solution: Set. $\alpha+\beta=K$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos K$$

$$\left(\frac{a}{R}\right)\left(\frac{b}{R}\right)-\left(\frac{\sqrt{R^2-a^2}}{R}\right)\left(\frac{\sqrt{R^2-b^2}}{R}\right)=\cos K$$

$$ab-\sqrt{(R^2-a^2)(R^2-b^2)}=R^2\cos K$$

$$\sqrt{(R^2-a^2)(R^2-b^2)}=ab-R^2\cos K$$

$$(R^2-a^2)(R^2-b^2)=a^2b^2-2abR^2\cos K+R^4\cos^2 K$$

$$R^4-(a^2+b^2)R^2+a^2b^2=a^2b^2-2abR^2\cos^2 K+R^4\cos^2 K$$

$$R^4-(a^2+b^2)R^2=-2abR^2\cos K+R^4\cos^2 K$$

$$R^2-(a^2+b^2)=-2ab\cos K+R^2\cos^2 K$$

$$R^2(1-\cos^2 K)=a^2+b^2-2ab\cos K$$

$$R^2-\frac{a^2+b^2-2ab\cos K}{\sin^2 K}$$

or $$R=\mathrm{D}\csc K \qquad \text{(Where } D \text{ is other diagonal)}$$

$$\theta=\tan^{-1}(\mathrm{D}\csc K)$$

$$\alpha=\cos^{-1}\left(\frac{a\sin K}{D}\right)$$

From: $R=D\csc K$ we arrive at the

Theorem: The diagonal joining two opposite right angles in a quadrilateral equals the product of the other diagonal and the sine of either of the other angles.

i. e. $$D=R\sin K$$

Note that K can equal 90° and the theorem is still valid.

EDITORIAL NOTE: This interesting application of mathematics to engineering is a special case of a Theorem of Ptolemy, Johnson's *Modern Geometry*, page 62. The theorem states, if quadralateral $ABCE$ is inscribed in a circle, then $BE\cdot AC=AB\cdot CE+AE\cdot BC$. Now let AC be a diameter and it follows that any chord equals the diameter times the sine of the subtended angle. In this case $BE=(AC)\operatorname{Sin} A$, or as above $D=R\operatorname{Sin} K$.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

Development of Mathematics in Scotland, 1669-1746*

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The fame of the MacGregor clan is often mentioned in the writings of Sir Walter Scott. That it was very powerful is suggested in the couplet:

> "*The moon's on the lake, and the*
> *mist's on the brae,*
> *And the clan has a name that*
> *is nameless by day.*"†

Most fortunate for the later descendents, the name was changed to Gregorie (later to Gregory) in 1603, and the edicts against the family of MacGregors could not longer be imposed upon the Gregories. It seems, however, that a change of name did not "obliterate the warlike character of their highland fathers, as is evident in the fact that as late as the 19th century it is reported that a certain James Gregory was accused of beating his fellow-professor in Edinburgh University quadrangle."‡

The fame of the Gregory family rests, not on any one genius, but rather on the brilliant men belonging to it who have been great teachers as well as investigators. "From the middle of the 17th century to the middle of the 19th century, some of the Gregorie connections were professing in one or another of the Scottish Universities."‡ During this period, the descendants of John and Janet Gregory occupied the chairs of Mathematics, Medicine, Chemistry, and History or Philosophy in almost an unbroken sequence; four chairs at St. Andrews, one at Glasgow, seven at Aberdeen, and ten at Edinburgh.

*See NATIONAL MATHEMATICS MAGAZINE, May, 1944.
†Scott, *The MacGregor's Gathering*.
‡Stewart, *The Academic Gregories*, p. 10.
‡Stewart, *Ibid.*, p. 10.

They were great teachers with a genius for lucid explanation often possessing the highest inventive powers and with gifts of leadership."

This family is mentioned by Galton in his book on *Hereditary Genius* as a striking example of hereditary scientific gifts. It is his opinion that this ability came into the family when Rev. John Gregorie, a parish minister, married Janet Anderson in 1621, since there is no record of special scientific power in the family up to this time. Galton states that "it may be taken for granted that the ability came from the Andersons, who were distinguished in foregoing generations."

With the exception of a brief period, the Chair of Mathematics in the University of Edinburgh was occupied by a member of the Gregory family from 1674 to 1725. The first of these, James Gregory, the third son of Janet and James "was certainly the cleverest member of this very illustrious family."* He received his early training in mathematics from his mother, and entered school far ahead of his class. He attended Marischol College, Aberdeen, from which institution he received his degree.

In those early days telescopes invented by Galileo were used by scientists. It was necessary to lengthen the tube to a hundred feet or more. By combining lenses and mirrors, Gregory showed that it was possible to obtain better results within a length of six feet or less. To the non-scientific person this instrument had great merit. "It presented objects in their true form, rather than inverted as is true of most telescopes."† In 1663, his book, *Optica Promota*, was published in London. It contained a masterly account of mirrors and lenses, and describes his telescope in detail. He was anxious to have his ideal constructed, and for this purpose he went to London. His book had been read with great interest by mathematicians everywhere, as well as by John Collins, the Secretary of the Royal Society. Collins introduced Gregory to the best opticians in London, but these failed to produce satisfactory lenses and mirrors, so that the project had to be abandoned. The attempt, however, had interested Hooke, a celebrated experimenter, who was able to succeed where others had failed, and in the 18th century "it became the standard astronomical pattern."‡ A beautiful instrument of this type may be seen, even to this day, in the Royal Observator of the city of Edinburgh, very little used at present, but as clear as the day it was made.

Meanwhile, Sir Isaac Newton, working independently at Cambridge also invented a reflecting telescope which was exhibited at the

*Stewart, *Ibid.*, p. 27.
†Turnbull, *James Gregory, Tercentenary Memorial Volume*, p. 3.
‡Turnbull, *Ibid.*, p. 4.

Royal Society in 1672, and brought fame to the maker. "It was but six inches long, and differed from Gregory's original design much as a flute differs from an oboe: the observer gazes directly into Gregory's and sideways into Newton's instrument."*

James Gregory spent the greater part of four years (1664-1668) in Padua, Italy. Here he met the great geometers of the Italian school. Stefano degli Angeli occupied the chair of Mathematics, and his teaching greatly influenced Gregory, especially as it gave him the fundamentals of the *Method of Tangents* and *Quadratures*, the foundations of differential and integral calculus respectively. Previous to this time he had used these same ideas in his *De Infinitis Parabolis*, applying them to curves of the form $y^n = b^{n-1}x$. The treatment was geometrical, and lacked analytic rigor. But it did give an insight into the mathematical ability of the young man, not yet 20 years of age.

The years in Italy greatly matured Gregory's mathematical ability as is shown by the fact that soon thereafter he produced two "imperishable little volumes," one called the *Vera Quadratura*, the true quadrature of the circle and the hyperbola, and the other the *Geometriæ Pars Universalis*, "which transformed the work of his master degli Angeli into a general philosophy."† These two books added greatly to Gregory's reputation as a brilliant thinker. As a result he was awarded a Fellowship in the Royal Society of London.

In the *Vera Quadratura*, Gregory attempted to prove the transcendence of π and e, a property of these two numbers not fully established until the time of Lindeman at the close of the 19th century. However, this book shows much originality, and contains many mathematical concepts such as convergence, functionality, difference between algebraic and transcendental functions, iteration processes, the similarity between circular and hyperbolic functions, as well as ideas concerning invariant functions.

As was often the case in the early years of the development of scientific ideas, the ingenuity and boldness of Gregory's thinking aroused a storm of opposition. A copy of *Vera Quadratura* was sent by Gregory to Huygens. Instead of the friendly suggestions and criticisms expected, Huygens accused him of using his own results without giving due credit. "Gregory retaliated by inserting in his next book, *Exercitationes*, a few pages which advanced the work of Huygens beyond all recognition." That this was done deliberately, is indicated in Gregory's own words, "I shall here try to bring the squaring of the circle and hyperbola to such perfection that Huygens will not recog-

*Turnbull, *Ibid.*, p. 4.
†Turnbull, *Ibid.*, p. 4.

nize his own off-spring."* It might also be noted that Sir Isaac Newton was pursuing the same line of thought.

A critical examination of Gregory's writings leads one to believe that he was conversant with the series for the expansion of $\sin x$, $\cos x$, and tan; $\sin x$ he obtained presumably by successive integration of $1-\cos x$, and $\tan x$ by dividing $\sin x$ by $\cos x$.

In spite of the fact that many mathematicians were extremely critical of everything that Gregory wrote, there were others who were sympathetic and helpful. John Collins, to whom Gregory applied for assistance when he was attempting to realize his ideas in the form of a telescope, became his life long friend, and aided him in many ways. As secretary of the Royal Society, Collins kept him well informed concerning problems which others had failed to solve, as well as the newest discoveries in scientific fields. One of these discoveries by Nichola Mercator was the important expansion of $\log(1+a)$ into the series

$$a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+\cdots.$$

Gregory at once acknowledge the fact that this was a much better series than his own, and is reported to have stated that "it was a hard matter in this age to write a book which should not presently be rendered naught."† In the correspondence which followed, Gregory answered a problem which had baffled Mercator and which cleared up a seventy-year old mystery conserning sea charts. Namely Gregory established the fact that $\int \sec x\, dx = \log(\text{sex}\, x+\tan x)$. This and many other properties of logarithms were included in *Exercitationes*, published in London in 1668 as a sequel to *Pars Universalis*.

In this same year, 1668, King Charles II was persuaded to establish a Chair of Mathematics in the University of St. Andrews. James Gregory was selected for the position and came as a stranger to that university, the first Professor of Mathematics. In a letter to his friend Collins he outlined his duties as follows: "I am now much taken up with my duties, and have been so all this winter, both with my public lectures which I have twice a week, and resolving doubts which some gentlemen and scholars propose to me."‡ During the six years he was at St. Andrews "he quietly made his discoveries, keeping in touch with the outside world through the letters of the faithful Collins."‡

*Turnbull, *Ibid.*, p. 6.
†Turnbull, *Ibid.*, p. 7.
‡Turnbull, *Ibid.*, p. 9.
‡Turnbull, *Ibid.*, p. 9.

In 1669 Gregory married Mary Burnet, the widow of John Burnet, who brought to the Gregory family a rich background of artistic taste, as well as an inherited scientific and literary ability. Never-the-less "the astronomer found love-making dreadfully time consuming and vaguely regretted it."* It was apt to interrupt his correspondence with such men as Huygens, Halley, Newton, and Collins. Quoting from one of his own letters— "I have several things in my head as yet only committed to memory, neither can I dispose of myself to write them in order and method till I have my mind free from other cares."†

At St. Andrews Gregory's interest centered in the field of astronomy. He lived near the beautiful cathedral, and there his name is still remembered in names of streets,—Gregorie's Lane and Gregorie's Place. The large upper room of the library was his study and workshop, very suitable for carrying out astronomical observations. It here that he laid plans for an observatory, the first of its kind in Great Britain. His influence and enthusiasm spread to such an extent that he gained the confidence of men of means, who provided the instruments. A pendulum clock, devised by him and made in London, is still in working order. This is one of the earliest clocks of its kind, following soon after Huygens had announced the laws of the pendulum.

The years at St. Andrews were very fruitful years for James Gregory. He discovered some of the effects of defraction grating. The year 1670 was decidedly full of activity, especially in mathematics. He discovered the binomial and interpolation theorems. It was during this year that he worked on the properties of the equiangular spiral. On the learning of this, Collins sent him an advanced sheet of Barrow's *Lectiones*, concerning this same curve. "A few weeks later, the book arrived at St. Andrews, and within a month, Gregory had poured out such a volley of equations in his next letter that Collins was convinced beyond all doubt that Gregory had made the same discovery of the calculus."‡ Perhaps the outstanding achievement of this period occurred February of the following year when he hit upon one of the most important theorems of all mathematics known as the Taylor Theorem. Unfortunately Gregory never published this crowning success of his mathematical investigations. When he received word that Sir Isaac Newton had anticipated him, he decided to withhold his own results until his co-worker had a chance to make the results known to the public. As this did not occur until several years after Gregory's death, many of his best discoveries were not made public during his life time.

*Stewart, *Ibid.*, p. 32.
†Stewart, *Ibid.*, p. 32.
‡Turnbull, *Ibid.*, p. 12.

In the summer of 1674 Gregory was offered a position in the Town College of Edinburgh, and became the first person to occupy the chair of Mathematics in this university. Why he left St. Andrews is best told in his own language in a letter to his friend Frazier in Paris. "I was ashamed to answer, the affairs of the observatory in St. Andrews were in such a bad condition; the reason of which was, a prejudice the masters of the university did take at Mathematics because some of their scholars, finding their courses opposed by what they had studied in mathematics, did mock at their masters, and deride some of them publicly. After this the servants of the college got orders not to wait on me at my observatory: my salary was also kept back from me: and scholars of most eminent rank were violently kept from me, contrary to their own and their parents' wills, the masters persuading them that their brains were not able to endure it. These and many other discouragements obliged me to accept a call here at Edinburgh, where my salary is nearly double, and my encouragements otherwise much greater."*

Evidently Edinburgh was more ready to receive the new ideas, for when Gregory arrived, he had a great reception. He was given freedom of thought and action, and he passed many happy hours in his observatory with his students, showing them the belted Saturn and the satellites of Jupiter. He became interested in solving quintic equations, and when a problem was sent by mathematicians in Paris to challenge the world, Gregory promptly solved it. Everything pointed to a happy and successful life in his new position. But in October of his first year, he was suddenly attacked by an illness while at the observatory with some of his students. This was followed by blindness, and soon after by death, at the age of 37.

A paragraph from the *James Gregory Tercentenary Volume*, edited by Herbert Western Turnbull, now Professor of Mathematics in the University of St. Andrews, states clearly the esteem and honor accorded this great scientist by Scottish Mathematicians of the present day. "The fiery MacGregor was sometimes impatient of criticism, stupidity and injustice, but deep in his heart lay that love of truth which led him to follow with disinterested devotion the bent of his genius and to rejoyce in the triumphs of others. He died at the height of his powers, leaving friends at home and abroad, the warmeh of whose affection and respect was expressed in words sometimes startling in their depth, friends from Italy, Flanders, France, Germany, England, and Scotland. It is proper to record that a July day in 1938, three hundred years after the year from which this story dates, there was gathered in the library at St. Andrews, where Gregory worked so long, a company of their

*Turnbull, *Ibid.*, p. 14.

mathematical descendants, distinguished guests from the world of science, from the Cambridge of Newton, the Paris of Cassini, the Germany of Leibniz, the Flanders of Huygens, and the New World of America, assembled in Scotland where mathematics is still pursued for its beauty and its truths."*

The old letters upon the backs of which James Gregory wrote his notes during the years at St. Andrews and Edinburgh fell into the hands of his nephew David, who succeeded him as Professor of Mathematics in the University of Edinburgh. The "Methodical David" listed all letters and papers written by his uncle, and they are now at the University of St. Andrews, having been bequeathed to that institution in very recent times.

During the years immediately following the death of James Gregory, before the appointment of David, an interim arrangement was made with a student, Mr. John Young, who carried on the work of the Mathematics Department with the title of Mathematical Tutor. At the age of 22, David Gregory assumed the responsibilities imposed upon him by his appointment as Professor of Mathematics in the University of Edinburgh. This occurred in 1683, just after he had passed the examination for the M.A. degree, "but previous to his laureation."

A manuscript volume of notes detailing the course of lectures given by Professor Gregory during his years at the University of Edinburgh was preserved by one of his students, Francis Pringle, afterwards Professor of Greek in the University of St. Andrews. The range of subjects prompted Professor Chrystal† to state that "the "the curriculum taught will bear comparison with our curriculum as it is now. There are lectures on Trigonometry, Logarithms, Practical Geometry, Geodesy, Optics, Dynamics and Mathematics."‡ While David Gregory was an excellent lecturer on Mathematics and Optics, yet he was distinguished primarily for his appreciation of Newton's ideas He was the first to give public lectures on the Newtonian Philosophy. This he did 35 years before these doctrines were accepted as part of the public instruction at Cambridge, Newton's own institution. "It was his object to bring the Principia down to the level of mathematical minds, and both he and his brother James who held the corresponding chair at St. Andrews were teaching Newton's philosophy before it was taught at Cambridge."‡‡

*Turnbull, *Ibid.*, p. 15.
†Professor of Mathematics, University of Edinburgh, 1879-1911.
‡Grant, *Story of the University of Edinburgh*, Vol. 2, p. 296.
‡‡Stewart, *Ibid.*, p. 54.

So enthusiastic was he about the *Principia* that he brought it to the attention not only of his students but also of Englishmen throughout Great Britain. Whiston, Professor of Mathematics at Cambridge, (1703-1707), says "that he was greatly excited to the study of Sir Isaac Newton's wonderful discoveries by a paper of Dr. Gregorie's when he was a Professor in Scotland, wherein he gave the most prodigious commendations to that work, as not only right but the effect of a plainly divine genius—while we at Cambridge, poor wretches, were ignominiously studying the fictitious hypotheses of the Cartesians."* Newton's own opinion of David Gregory is recorded in a letter to Florinstead, one of his contemporaries. Referring to their common scientific work he sayd, "If you and I live not long enough, Mr. Gregory and Mr. Halley are both young men."†

In 1684 Professor Gregory produced his first work entitled *Exercitatio Geometrica de Dimensione Figurarum,* which contained some posthumous papers written by his uncle, with additions of his own. The book was not widely read but it gave "A public proof of his compentancy to discharge the duties of the important position to which he had been appointed."‡

David Gregory received his appointment during the reign of Charles II, but changes had come about, and William and Mary were on the throne. An investigation was undertaken to ascertain the political opinions of all men to whom was intrusted the instruction of the youth. A large commission was appointed to deal with the Universities of Scotland. A meeting of the Commission was held in Edinburgh in July, 1690. When Gregory's turn came, "he like others was accused by men of whose names he was kept in ignorance, and whose statements he could not but feel were libelous, malicious and false."‡ The lay members were inclined to favor him, and when they inquired into his conduct as a teacher, he was able to present an excellent report. At the same time he would not subscribe to the Confessions of Faith. For this reason he could not be assured that his possession of the Chair of Mathematics would be continued; neither could his supporters at the time of the trial give him any assurance on this point. However, the records show that "Dr. David Gregory, the only truly great man among the Episcopalian Professors was wisely spared."§ But the suspense was intolerable, and he was glad of an opportunity to quit the institution in which he had received so

*Grant, *Ibid.*, Vol. 2, p. 296.
†Grant, *Ibid.*, Vol. 2, p. 297.
‡Stewart, *Ibid.*, p. 56.
‡Stewart, *Ibid.*, p. 52.
§Stewart, *Ibid.*, p. 58.

much annoyance. During the nine years in Edinburgh he had "brought the mathematical teaching into the vanguard of scientific progress."*

Gregory's fame in England was so great that in 1692 he was elected Fellow in the Royal Society. Encouraged by this fact, together with the cordial treatment he had received from many of the English Mathematicians, he became a candidate for the Professorship of Astronomy at Oxford. The only other candidate was Edmund Halley, whose claims from a scientific point of view were equal, if not superior, to those of Gregory. But Halley was an infidel, and this disqualified him in the eyes of the patrons of the chair. As a result David Gregory, supported by a strong recommendation from Sir Isaac Newton, received his appointment.

Not long after Gregory came to Oxford, he began to contribute to the volumes of the Royal Society. He sent in a beautiful solution of the Florentive problem which Viviani had sent as a challenge to British Mathematicians. "His solution was masterly and delighted geometers."† During his early years in Oxford he found much more time to write than he had found in Scotland, where teaching always had to come first. He studied the properties of the catenary, and was the first to observe that by inverting this curve, the legitimate form of an arch results. Shortly after his marriage in 1695 he brought out his book on *Catopricæ et Dioptricæ Sphericæ Elementa*, a great work on looking glasses and lenses. "He was rewarded with praise, his book was promised immortality."‡ In this book a suggestion was made that "Nature does nothing in vain, and in imitation of nature the object glasses of telescopes might well be composed of media of different density."‡ After the invention of achromatic glasses by John Dollond in 1758, the meaning of this suggestion was clear.

Life in Oxford for Gregory finally turned out to be quite different from his anticipations. In fact he committed the only sin which Dickens calls unpardonable—he was too successful. The time came when he was greatly criticised by other members of the faculty, "possibly more than otherwise have been the case, because he was so well known to the outside world."§ If, however, he was not universally appreciated at the University, he was in great favor at the Court, since he was appointed mathematical preceptor to Princess Ann's son, the young Duke of Gloucester.

He busied himself with suggestions concerning curriculum changes. Among others, he proposed that all teaching should be done in English,

*Grant, *Ibid.*, Vol. 2, p. 297.
†Stewart, *Ibid.*, p. 61.
‡Stewart, *Ibid.*, p. 62.
‡Stewart, *Ibid.*, p. 64.
§Stewart, *Ibid.*, p. 63.

and that undergraduates should study Euclid, trigonometry, algebra, mechanics, catoptrics, dioptrics, astronomy, and the theory of navigation. "The teacher," he said, "should always be ready to gratify the request of those who desire his instruction. And if any students were found hungering and thirsting, they were to be given regular demonstrations of the operations of integers and fractions, vulgar or decimal when they pleased"* As to the proper number for a class he suggested that there should be not fewer than 10, and not more than 20. Whether these proposals became effective at that time is uncertain, but it did serve to bring him into close contact with the authorities of the university.

When negotiations for the union between Scotland and England began, David Gregory together with Paterson, the founder of the Bank of England, was appointed to decide upon "What equivalent was to be paid to Scotland for bearing her share of the debt of England, which, of course, was afterwards to be considered as the debt of England."† There was much discussion on this particular article of the treaty of union, but those in authority were absolutely satisfied with the manner in which the work was done.

David Gregory died in 1710 at the age of 49. After his death, Colin Maclaurin‡ published an English translation of Gregory's work, *A Treatise on Practical Geometry*, the first edition of which was sold in a short time, as this book was used in its day as a text in all of the Scottish universities.

On the departure of David Gregory to Oxford, his brother James was appointed Professor of Mathematics in the University of Edinburgh. He held this position for 33 years, until 1725. "He seems to have been a very good teacher, but otherwise he did not add to the reputation of the Gregory family."⁑ At the early age of 20 he was appointed to the Chair of Philosophy in the University of St. Andrews. Like his elder brother he was very much in advance of his age, and like him, became an outstanding exponent of the Newtonian Philosophy. He wrote a thesis consisting of 25 propositions, the majority of which are found in the *Principia*, followed by three relating to logic and the use of it in the Aristotelian and Cartesian Philosophies. In this thesis he defines logic as "the art of making a proper use of things granted in order to find what is sought."§

Professor Gregory occupied the Chair of Philosophy at St. Andrews until the outbreak of the great revolution. His love for the

*Stewart, *Ibid.*, p. 66.
†Stewart, *Ibid.*, p. 71.
‡Professor of Mathematics, University of Edinburgh from 1724 to 1746.
⁑Grant, *Ibid.*, Vol. 2, p. 298.
§Stewart, *Ibid.*, p. 89.

dethroned king, Charles II, caused him to resign. He could not bring himself to take the oath of allegiance to William and Mary, and so for awhile he was without a position. However, when his brother David resigned his position at the University of Edinburgh, feeling was no longer bitter, and he succeeded to the chair in 1692. In 1725 he was superannuated because of ill health, and Mr. Colin McLaurin was made joint Professor. Thus "the ablest in every respect of all the occupants of the Mathematical Chair, if he except the first Gregory (whose career in Edinburgh was too brief for him to be brought into comparison) commenced his course in the University of Edinburgh."*

Colin McLaurin was born in 1698. At eleven years of age he entered the University of Glasgow, and a year later he showed his mathematical ability by mastering in a very brief period of time the first six books of Euclid, a copy of which has been loaned to him by a friend. After graduation at the age of 15 he read mathematics and natural philosophy for a period of three years. At the age of 19 he was elected Professor of Mathematics in Marischal College. Two years later "he was made Fellow of the Royal Society, brought out papers in their *Transactions*, published his *Geometrica Organica*, and made the acquaintance and became the favorite of Sir Isaac Newton. He left the position at Aberdeen to travel abroad, and while in France he wrote a tract *On the Percussion of Bodies*, for which he was granted the Prize of the French Academy of Science. It was on his return from this trip that he was chosen to be joint Professor with James Gregory.†

"McLaurin's discoveries in the department of pure mathematics are of the most exquisite beauty; and at the same time he shows in his application of mathematics to physical problems that power of seizing the vitally important amidst a mass of irrelevant details."‡ His chief work, published during his lifetime, was a treatise on *Fluxions* (1742) in two volumes, in which appears the theorem on expanding a function into a series for which he is so widely known. He also wrote an essay on the tides which shared with Euler and Daniel Bernoulli the French Academy prize. For the benefit of his wife and children his executors published his posthumous works, namely his Algebra, and his account of Sir Isaac Newton's Philosophy.

McLaurin was not only a great scientist and educator, but he made himself felt in the social and economic life of the city, and soon became the "life and soul"‡ of the University. He was a prominent

*Grant, *Ibid.*, Vol. 2, p. 299.
†Grant, *Ibid.*, Vol. 2, p. 299.
‡Grant, *Ibid.*, Vol. 2, p. 201.
‡Grant, *Ibid.*, Vol. 2, p. 299.

figure in all of the scientific circles of Edinburgh. He was made Secretary of the Society for improving Medical Science, and became the editor of its journal. He was influential in the expansion of this organization, which finally became the Royal Society of Edinburgh. To him is due a law providing a fund for the widows of ministers and professors of Scotland, for which he made the acturial computations, thus placing the Assurance Fund on a secure foundation. In 1739 he drew up directions for the survey of the Orkney and Shetland Islands, which was carried out by his own students.

In 1745 when the Jacobite army was advancing toward the city, he was selected to act as military engineer and to fortify the city. He was never a strong man physically, and the anxiety and fatigue to which he was exposed "laid the foundation of the distemper which proved fatal to him."* His eulogy was pronounced in the university by Alexander Monro, primus, who stated that "acute parts and extensive learning were in Mr. McLaurin but secondary qualities; and that he was still more nobly distinguished from the bulk of mankind by the qualities of his heart, his sincere love of God and men, his universal benevolence and unaffected piety, together with constancy in his friendships that was in a manner peculiar to himself."†

A study of the Development of Mathematics in Scotland reveals the fact that very little was done in this field until the later part of the 17th century. Then the influence of the three Gregories and Colin McLaurin as Professors of Mathematics in the University of Edinburgh brought Mathematics to a very high level in a short period of time. This may be seen from the rich program of McLaurin's printed in *Scots Magazine* in 1741. He gave "every year three Colleges, and sometimes a fourth upon such of the abstruse parts of the Science as were not explained in the former three. The first course contained: Demonstrations of the ground of Vulgar and Decimal Arithmetic; six books of Euclid; Plane Trigonometry and use of Logarithms, Sines, etc.; Surveying, Algebra; and a lecture on Geography once a fortnight.

The second course consists of: Algebra; the Theory and Mensuration of Solids; Spherical Trigonometry, the doctrine of the Sphere, Drilling and other practical parts; Conic Sections, with the theory of Gunnery; the elements of Astronomy and Optics. He begins the third College (continues the *Scots Magazine*) with Perspective; then treats more fully astronomy and Optics. Afterwards he prelects on Sir Isaac Newton's *Principia*, and explains the direct and inverse methods of Fluxions. At a separate hour he begins a College of Ex-

*Grant, *Ibid.*, Vol. 2, p. 300.
†Grant, *Ibid.*, Vol. 2, p. 300.
‡Grant, *Ibid.*, Vol. 1, p. 271.

perimental Philosophy about the middle of December which continues thrice every week till the beginning of April; and at proper hours of the night he describes the constellations, and shows the planets by telescopes of various kinds."* All this busy teaching of important and interesting subjects was comprised in the time between the first of November and the first of May, thus leaving the Professor six months in the year for his own researches.

BIBLIOGRAPHY

Grant, Sir Alexander, *The Story of the University of Edinburgh*, Vols. I and II.
Stewart, Agnes Grainger, *The Academic Gregories.*
Turnbull, Herbert Western, *James Gregory—Tercentenary Memorial Volume.*
Smith, David Eugene, *History of Mathematics*, Vols. I and II.
Dalzer, *History of the University of Edinburgh*, Vols. I and II.
Scott, *The MacGregor's Gathering.*
Wilson, *The History of Mathematics in Scotland.*
Dixon, *The Edinburgh Book of Scottish Verse.*
Dictionary of National Biography.
Encyclopaedia Britannica.

To Our Subscribers

The next four numbers of NATIONAL MATHEMATICS MAGAZINE *will be printed by Franklin Pres, in as quick succession as may be possible. Indeed the February issue should be off the press not long after the January number has been mailed out. Most of the materials for March, April and May are at hand, and every effort of our printers will be made to put the remaining numbers of Volume XIX into our subscribers' hands before much of the summer has gone by.*

We are grateful—deeply so—for your patience and forbearance during these long periods between issues. The causes of these delays, seemingly insurmountable for a long time, seem at last to have been overcome.

Please help the central office in Mobile by forwarding your renewal checks without waiting for statements.

S. T. SANDERS.

The Teachers' Department

Edited by
WM. L. SCHAAF, JOSEPH SEIDLIN, L. J. ADAMS, C. N. SHUSTER

Teaching the Calculus

By R. C. DRAGOO
University of Oklahoma

*Introduction.** It is the opinion of the writer that elementary calculus should be taught in such a way that meaning and use would precede the proofs of the various topics as they are covered in the course. It is also believed that meanings often are not stressed as they should be in classroom discussions or in the textbooks generally in use. It is the purpose of this paper to discuss the importance of meanings and to outline briefly some means by which the meaning of functions, derivatives, differentials, and integrals may be presented to the student.

Proofs and Meanings. A fairly thorough search has revealed that little has been written on the teaching of the calculus, but the literature that is found brings to the attention of the reader at once the fact that there is a diversity of opinion on the matter of the place of rigorous proofs in teaching elementary calculus.

Some writers are of the opinion that very thorough proofs should be mastered before anything is said about their use and that meanings should come with the proofs. A proponent of this idea is R. L. Goodstein(1) who says that the calculus course as it is usually taught is a compromise, it being thought generally that rigorous proofs are beyond the understanding of the students for whom the course is planned. He believes that it is expecting too much of the student to ask him to understand the concepts of the theory of functions when no use is made of the real number theory upon which the theory of functions is based, and declares that we as teachers "must endeavor to hide the deficiencies of our reasoning beneath the kindly cloak of geometrical intuition." Mr. Goodstein suggests a new approach to the subject which perhaps is effective for the brilliant student, but which, I believe,

*Read before the mathematics colloquium at the University of Oklahoma, January, 1944.

would discourage the average student of the calculus. It seems that the thing to be desired in any course is the immediate interest of the student and that a very poor way to get that interest is by beginning with a mass of involved abstract reasoning in which the average student can see little or no meaning. It is better to explain the meaning, let the student use his intuition, tie the thing up with ideas he has used previously, use it in working out some practical problems and then proceed to the proof.

The other extreme, it seems to me, is implied in a book by S. P. Thompson(2) in which practically no proofs are given. However, there is something in his statement, "you don't forbid the use of a watch to every person who does not know how to make one. You don't object to the musician playing on a violin that he himself has not constructed. You don't teach the rules of syntax to children until they have already become fluent in the use of speech. It would be equally absurd to require general rigid demonstrations to be expounded to beginners in the calculus."

The idea that more nearly conforms to my opinion in this connection is contained in the following statement by Edward V. Huntington.(3) "Every mathematical theorem presents itself to the student in a dual capacity. It is something to be used in later work, and it is something to be proved by logical demonstration. Neither of these two aspects should be ignored ... we must teach the student how to use the powerful theorems of the calculus, and how to convince himself and others that these theorems are true." Mr. Huntington indicates that what he says refers to students of physics, chemistry, biology, and mathematics itself. He is of the opinion that the engineer needs the rigorous proofs in order to understand the limitations of the theorems he uses in the working of new problems which he will encounter in his profession; also, that the student of pure mathematics needs the stimulus he can get through contacts with the practical uses of the theorems which he uses.

It is believed that most of us will agree with Mr. Huntington in the equality of value of these two aspects, and I, personally, agree with him when he says he believes that in many cases the proof of a theorem should come after, rather than before, the use of that theorem in practical applications. In many cases, it is only after the meaning of a theorem has become thoroughly familiar through use that its proof makes any impression. By putting use first and proof second in order of sequence, it is believed that a saving of time can be affected without loss of mathematical perspective.

The presentation of meanings and use before proof is the logical and historical sequence, particularly as applied to elementary mathe-

matics. First there is the need, the satisfying of the need through operations involving previous experience, which more often than not involves intuitive reasoning, and then the proof of the validity of the operations used. Most mathematics has been developed in this order. After the generalization has been developed, a multitude of new applications often present themselves. One great value of thorough proofs is that they make the student more fully aware of the possibilities as well as the limitations of the theorems he uses.

With reference to the development of mathematics, Felix Klein[4] says, "The investigator himself ... does not work in a rigorous deductive fashion. On the contrary, he makes essential use of his phantasy and proceeds inductively, aided by heuristic expedients. One can give numerous examples of mathematicians who have discovered theorems of the greatest importance but which they were unable to prove." Klein further says that it is in the discovery and in the development of the infinitesimal calculus that the inductive process, built up without compelling logical steps, played such a great role. He tells how some of the ideas of the calculus were developed, and cites specific examples showing that intuition played a very great part in it.

An examination of late textbooks in calculus shows a very definite attempt to stress the idea of meanings from the very beginning. Reviewers of textbooks almost invariably make reference to the simplicity and understanding with which the subject is presented. However, there are some points at which, it seems to me, most writers of textbooks err in this respect. Let me point out one instance. In many of the present day textbooks, the authors, in proving the Theorem of Mean Value, start out by saying, "From the function

$$\Phi(x)=\frac{f(b)-f(a)}{F(b)-F(a)}\,[F(x)-F(a)]-[f(x)-f(a)]\text{''}.$$

No attempt is made to give the student any idea as to where the function comes from, or the reason for which it was formed, save that it satisfies the conditions of Rolle's Theorem. How much better it would be to show first the meaning of the theorem, how it is to be needed in later work, and then proceed to the proof of the simpler theorem

$$\frac{f(b)-f(a)}{b-a}=f'(x_1),$$

where $f'(x_1)$ is the derivative of $f(x)$ at some point x_1 between a and b. Since the more general Theorem of Mean Value is not necessary in elementary calculus, it is suggested that it not be presented here. The rigor and extent of any proof should be compatible with the need of the student. First he should be convinced of the need, and then

the thoroughness and completeness of the proof should be determined by that need. Very excellent methods of presenting the Theorem of Mean Value are presented by A. A. Bennett[5] and Neeley and Tracey.[6] There is no justification for hiding reasons and meanings, particularly at this stage of mathematical learning, when they are so readily available.

The Function. Since the concept of function is found in so much of our mathematics, and as students seem to have so much difficulty in grasping the idea, it is necessary to stress the idea throughout the course in calculus. It is not sufficient to have the students work a few examples such as the following. Given $f(x)=x^3+x^2-2x+1$, find $f(x+h)$. It is necessary to go back to some very elementary illustrations. Familiar examples from geometry, physics, etc., may be employed to great advantage here. For example: $s=1/2\,gt^2$; $C=2\pi r$; $V=4/3\,\pi r^3$; $A=\pi r^2$; $pv=c$; etc. Show in each case how the value of the dependent variable is determined by the value assigned the independent variable, and why it is that the term "function" is used.

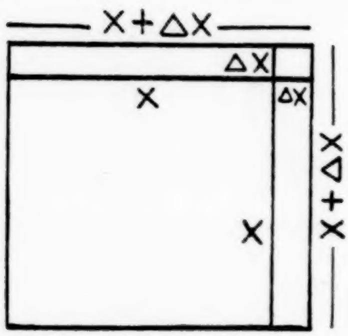
It is valuable to precede all of this by pointing out that the idea of function pervades a very great portion of our thinking. For example, the velocity of an automobile is a function of the pressure upon the accelerator; interest earned on money is a function of the rate and time; money earned on a job is a function of time; learning is a function of attention; the height of a child is a function of its age. The primary concern is that the student get the feeling of the meaning of function. If these fundamental concepts are fully appreciated, many of our later problems will be much easier.

It is surprising to learn that many of our students of calculus do not know why a function of an angle is called a "function." It is not sufficient for many students for the instructor to say that it is called a function of the angle because it depends upon the magnitude of the angle. It is very helpful here to construct a right triangle and point out the correspondence of the angle and the ratios of the various sides of the triangle. Trigonometric tables may also be used to show the relationship between the angle and the functions.

E. R. Hedrick[7] says, "The function concept should be stressed in every course. The function concept is the one theme which tends to unify all mathematics and to permit its integration with life and with science." It may seem too much to stress it in all courses, for it does seem that the idea should be grasped after a while. Yet experience teaches us that it is a concept that is difficult for some to comprehend, and that many students get lost in a maze of $f(x)$'s, $\Phi(x)$'s, etc.

The Derivative. We are all aware of the fact that if the derivative is presented as a formalized thing, rules can be learned and numerous examples worked without any comprehension of the real meaning. A good way to approach this idea is by calling the student's attention again to the meaning of the function and to how the dependent variable changes in value as the independent variable changes. Then take a specific example, such as $y=x^2$. Sketch the curve and call attention to the fact that as x increases, y increases (or decreases), and that from the curve it can be seen that the rate of change of y with respect to x at any two different points is different, and state that we call the rate of change in y with respect to the change in x, at a given point, the derivative of the function at that point. The usual explanation of the meanings of $\Delta y/\Delta x$ and dy/dx should be stressed. It should be pointed out that $\Delta y/\Delta x$ is the average rate of change in the dependent variable with respect to a corresponding change in the independent variable, while dy/dx is the instantaneous rate of change.

An analytic explanation of the derivative is as follows. In the function $y=x^2$, x may be thought of as the side of a square and y as the area. Draw a square x units on a side and then draw the square $(x+\Delta x)$ units on a side.



Now the area of the large square is $(x+\Delta x)^2$ and that of the small square is x^2. As the side changes from x to $x+\Delta x$, the area changes from y to $y+\Delta y$; so that $y+\Delta y=x^2+2x\Delta x+\Delta x^2$, and the difference between the large and small areas is Δy which equals $2x\Delta x+\Delta x^2$. Now the thing we are interested in is the ratio $\Delta y/\Delta x$, for this is that which shows the ratio of the change in y with respect to a corresponding change in x. So since $\Delta y=2x\Delta x+\Delta x^2$, $(\Delta y/\Delta x)=2x+\Delta x$. Now $\Delta y/\Delta x$ is the average rate of change of y with respect to x, and the derivative is the instanteous rate of change of y with respect to x at point P. That is, when Δx, Δy approach zero, $\Delta y/\Delta x$ becomes what we call dy/dx, and this equals $2x$. It should be pointed out to the

student that this value dy/dx is different for different sizes of the square. Some other examples should be given here such as $xy=c$, or $pv=c$. Most of the students will have happened upon Boyle's law somewhere, and it affords an excellent example for describing the rate of change of one variable with respect to a corresponding change in the other.

This is a very good place to work a lot of problems in maxima and minima so that the student can see that he has learned something of real value. For such problems, mere exercises will not suffice. Problems such as the following should be worked. Given the path of a projectile, find its highest point; find the minimum of material that can be used in a cylindrical can of say one quart volume; find the maximum area that can be included in a field with a given amount of fencing.

The Differential. There are those who think that the meaning of differential should be taught along with the idea of the derivative. Among these are E. V. Huntington(3) and E. G. Phillips.(8) It probably should be introduced earlier in the course than it is in many textbooks. Its relationship is so close to the derivative that it should certainly follow closely after it. Huntington(3) is of the opinion that the derivative and differential should be introduced together at the beginning of the course. He says that differential notation provides the technique that will be used exclusively after the course gets underway, that it is a waste of time to develop a technique which is to be abandoned in a few weeks, and that the reason for the postponement of the teaching of the differential is that it is a mysterious metaphysical entity that must be approached with fear and trembling. He is of the opinion that the derivative and the differential should both be introduced on the very first day of the course, and gives some illustrations of how this may be done by the use of specific examples. One of his examples is very ingenious and practical. It is suggested that most calculus teachers would benefit by reading this particular portion of his paper.

Most textbooks have very excellent graphical representations of the differential. These should be explained thoroughly to the student, simultaneously with the analytical explanation. For this purpose, I know of no better example than the function $A=x^2$, when A is the area of a square and x is the side. The whole idea of both differential and derivative can be adequately explained by its use.

The Definite Integral. Once the student has properly grasped the concepts of derivative and differential, probably not much difficulty will be encountered in understanding the integral. The idea of the definite integral is very well presented in most textbooks, particularly

where integration is considered as a process of summation. It is better understood, geometrically, however, if we consider it both as an area and as the difference between the ordinates of the integral at the points a and b, where these are the limits used. It is also helpful in this connection to sketch on the same axes of coordinates the integral and the integrand. I shall now illustrate this method of which most calculus teachers are probably aware.

We define the definite integral

$$y=\int_a^b f(x)dx \text{ as } F(b)-F(a), \quad \text{where } F(x)=\int f(x)dx,$$

and also as the area under the curve from $x=a$ to $x=b$. This first definition might be stated as the difference between the ordinates of the integral at $x=a$ and $x=b$. Since we are interested only in the difference of the ordinates, we may consider the constant of integration equal to 0, that is, $y=f(x)dx=F(x)+0$. Let us take first the simple example:

$$\int_1^4 (2x-5)dx=x^2-5x\Big]_1^4=0.$$

The result is zero, although there is obviously an area bounded. One answer is that equal parts of the area have different algebraic signs, and when added the result is zero. Another way to look at it is that it that it happens that the difference of the ordinates of the integral is zero. A general conclusion that may be drawn from this is that the value $F(b)-F(a)$ is not the true value of the integral if at any point between a and b the integral changes from an increasing function to a decreasing function, or the other way around. Or, another way to look at it is that the integrand must not vanish between a and b.

The concept of the value of the definite integral as the difference between its ordinates at $x=a$ and $x=b$ where these are the limits of integration brings out some interesting things concerning integrals which are inverse trigonometric functions. Consider the integral

$$\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}}=\text{arc sin}\frac{x}{2}\Big]_{-2}^{2}=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi.$$

Since $y=\text{arc sin}(x/2)$ is a multivalued function, students frequently ask, "Why not evaluate

$$\text{arc sin}\frac{x}{2}\Big]_{-2}^{2}=\frac{5\pi}{2}-\left(-\frac{5\pi}{2}\right)=5\pi?\text{"}$$

Of course, since the function $y = \text{arc}\sin(x/2)$ is multi-valued, we restrict the angle y to the pricipal values from $-\pi/2$ to $\pi/2$ for which the function is single valued. Examination of the graphs of the integrand and integral on the same coordinate axes will further convince students that the reasonable value is π.

BIBLIOGRAPHY

(1) Goodstein, R. L., "Teaching the Calculus", *Mathematical Gazette.* October, 1941.

(2) Thompson, Silvanus P., *Calculus Made Easy.* Macmillan Co., 1934.

(3) Huntington, Edward V., "Teaching the Calculus", from *Four Papers on the Teaching of Mathematics* by the Society for the Promotion of Engineering Education. April, 1932.

(4) Klein, Felix, *Elementary Mathematics from an Advanced Standpoint.* Macmillan Co., 1932.

(5) Bennett, A. A., "The Consequences of Rolle's Theorem", *American Mathematical Monthly.* January, 1924.

(6) Neeley-Tracey, *Differential and Integral Calculus.* Macmillan Co., 1932.

(7) Hedrick, E. R., "The Function Concept in Elementary and in Advanced Mathematics", *American Mathematical Monthly.* September, 1938.

(8) Phillips, E. G., "The Teaching of Differentials", *Mathematical Gazette.* July, 1931.

(9) Dockeroy, N. R. C., "The Teaching of Mathematical Analysis", *Mathematical Gazette.* December, 1935.

(10) Phillips, E. G., "Differentiation from First Principles", *Mathematical Gazette,* October, 1938.

(11) Temple, E., "The Rehabilitation of Differentials", *Mathematical Gazette,* May, 1936.

(12) Bell, E. T., "The Function Concept", *American Mathematical Monthly.* October, 1934.

(13) Cowbridge, F. T., "The Teaching of Differentials", *Mathematical Gazette.* February, 1932.

Post-War Blueprint

By H. L. Dorwart
Washington and Jefferson College

Those instructors who have been concerned over the lack of any knowledge of fundamentals on the part of some of their students may be interested in the following portions of an editorial from a recent issue of *School and Society*.

> A modified form of the "promotion-on-schedule" policy has been put into effect in the New York City public schools, according to a report of the Board of Superintendents that was recently made available. This change, the report emphasizes, does not mean 100-percent promotion. Some pupils, it seems, still repeat grades or half-grades. The proportion is not large, however, and whenever a pupil is "too mature socially" to be held back, "serious deficiencies in the fundamentals" do not stand in the way of his "advancement" to a higher grade.

> The public-school system of the metropolis, because of its magnitude and the consequent complexity and relative inelasticity of its administration, has generally been among the last to adopt far-reaching "reforms".

> Obviously, the problems that have confronted the high schools in their now practically complete change from a selective to a universal status have already affected the higher institutions—and will assume a far greater intensity and far more serious proportions if the present trend toward something akin to universal higher education continues.

After the war, mathematics in liberal-arts colleges may or may not be compulsory for all students. If it is not compulsory, then mathematics instructors probably will not be greatly bothered by the students who have been "passed" from grade to grade in spite of their lack of knowledge of elementary mathematics. But if the memory of the pre-Pearl Harbor mathematical unpreparedness of masses of college students remains until after the war, it seems likely that there may be some kind of a requirement in mathematics. Much depends on the form that this requirement takes. If, without taking into consideration the ability, the previous preparation, and the future needs of students, all are to be forced into the same mould, then nothing will have been learned from the past experience, and the requirement will not last long.

The following suggestion is certainly not new,* and probably is in use in many colleges. However, in view of the importance of the

*See *A Program in Freshman Mathematics* by E. A. Cameron, *American Mathematical Monthly*, Vol. 47 (1940), pages 471-473, and Comments on the North Carolina Program by H. L. Dorwart, *ibid.*, Vol. 48 (1941), pages 37-39.

problem, it seems worth while to have a restatement. To meet all needs, there should be a two-way offering of courses for the beginning college student—different subject matter for those entering different programs, and different levels of difficulty or development to take care of differences in student ability or previous preparation.

The following is a suggested minimum offering of five courses for a liberal-arts college:

Program / Student	Pre-business, pre-law, and social science majors.	Pre-engineering, pre-medical and science majors	Pre-ministerial, and "humanities" majors.
Poor or poorly prepared	*Course A* Intermediate high school algebra and review of principal theorems of plane geometry.		
Average	*Course B_1* Business Mathematics	*Course B_2* Algebra and Trigonometry	*Course B_3* General Cultural
Advanced		*Course C* Analytic Geometry and Introduction to Calculus	

Placement tests, together with a study of the amount and quality of previous mathematics, would determine the students who should take Course A. Ordinarily this course would carry no college credit. However, if it were to be taught to mature people (e. g. returning veterans), or from a mature point of view,* college credit might be given.

Courses B_2 and C would be the usual courses in these subjects, and they would probably have the major student registration. Adequate publicity should be given the fact that these courses are prerequisite to further work in mathematics and science.

Courses B_1 and B_3 would be terminal courses for the indicated groups of students. Course B_1 might well include such topics as progressions, binomial theorem, logarithms, mathematics of finance, and elementary statistics.

Course B_3, which would emphasize ideas rather than techniques, is still in the experimental stage. A recent study of general mathe-

*See description of Mathematics Courses 1 and 2 of the College of the University of Chicago, in recent catalogues of the University.

matics courses* should be consulted in this connection. Just before the war, such courses had many advocates, and, to quote Brown, "many of the instructors interviewed felt confident that after the national emergency the movement for cultural mathematics will be renewed with great emphasis." One of the major aims of this course would be to have the student carry with him into later life a sympathetic point of view toward mathematics, as opposed to the violently antagonistic attitude of many present-day adults who are products of the old-time, rigid, drill courses.

**General Mathematics in American Colleges*, by K. E. Brown, Teachers' College, Columbia University, 1943. (See also, *Mathematics for Students of the Humanities*, by Oystein Ore, American Mathematical Monthly, Vol. LI, October, 1944, pp. 453-458, published after the present paper was written.)

Brief Notes and Comments

Edited by
MARION E. STARK

12. *Perspective Triangles.* The interesting property of perspective triangles recently recalled by J. H. Butchart (this MAGAZINE, Vol. XIX, No. 1, October, 1944, p. 37, note 8)may be stated as follows:

Consider three triangles I, II, III such that I is circumscribed about II, and II is circumscribed about III. If the triangle II is perspective to the other two, the latter are perspective to each other.

This proposition may be generalized to read: *If any one of the three triangles is perspective to the other two, the latter are perspective to each other.*

Let $I=ABC$, $II=DEF$, and $III=KLM$, so that D, E, F lie on BC, CA, AB, respectively, and K, L, M, on EF, FD, DE, respectively. Let the lines AK, BL, CM meet the respective sides of I in the points X, Y, Z, and let BC, EF meet in H. Applying Menelaus' theorem to each of the triangles HBF, HCE and the transversal AKX, we have, both in magnitude and in sign,

$$\frac{HX}{XB}\cdot\frac{BA}{AF}\cdot\frac{FK}{KH}=-1 \qquad \frac{CX}{XH}\cdot\frac{HK}{KE}\cdot\frac{EA}{AC}=-1$$

hence, multiplying and simplifying,

$$\frac{AB}{AC}\cdot\frac{CX}{BX}\cdot\frac{AE}{AF}\cdot\frac{FK}{EK}=1.$$

Considering the other two pairs of corresponding sides of I and II, namely CA and FD, AB and DE, we obtain two analogous relations

$$\frac{BC}{BA}\cdot\frac{AY}{CY}\cdot\frac{BF}{BD}\cdot\frac{DL}{FL}=-1 \qquad \frac{CA}{CB}\cdot\frac{BZ}{AZ}\cdot\frac{CD}{CE}\cdot\frac{EM}{DM}=1.$$

Observe that these last two relations may be derived from the first by circular permutations.

If the three equalities are multiplied, the result, after simplification, may be put in the following form

$$(F) \qquad \left(\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}\right)\cdot\left(\frac{DL}{LF}\cdot\frac{FK}{KE}\cdot\frac{EM}{MD}\right)$$

$$\cdot\left(\frac{AY}{YC}\cdot\frac{CX}{XB}\cdot\frac{BZ}{ZA}\right)=1.$$

The points K, L, M lie on the lines AX, BY, CZ, by construction; hence the two triangles I and III are perspective, if and only if the last parenthesis in the formula (F) is equal to unity, by Ceva's theorem. Similarly, triangles I and II are perspectives, if and only if the first parenthesis is equal to unity, and the same thing holds for triangles II, III and the second parenthesis. Formula (F) shows that if two of the three parenthesis are equal to unity, the same is true of the third. Hence the proposition is proved.

University of Oklahoma. N. A. COURT.

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to E. P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 388. Proposed by *Harold S. Grant*, Rutgers University.

Show how to obtain the necessary and sufficient conditions that a polynomial equation of degree n possess a root of multiplicity $m(2 \leqq m \leqq n)$ in terms of rational, integral functions of the coefficients.

Revised solution by the *Proposer*.

The former solution (December, 1941, p. 158) is not entirely adequate since it assumes the non-vanishing of certain coefficients in carrying out the algorithm for finding the greatest common divisor. In fact, the problem, as originally proposed, needs modification in some way. Even in the simplest case, the vanishing of the discriminant does not insure that $f(x)=0$ possess a double root. It only implies that $f(x)=0$ has at least one root of multiplicity greater than 1—a much less specific requirement.

We therefore modify the problem to read as follows: Determine necessary and sufficient conditions that $f(x)=0$ have s_k k-fold roots, $1 \leqq k \leqq n$. Since the degree of $f(x)=0$ is n, we have

$$n=\sum_{k=1}^{n} ks_k. \qquad \text{we note:}$$

(1) $f(x)$ and $f'(x)$ have a G.C.D. of degree $n_1=\sum_{k=1}^{n}(k-1)s_k$, say $d_1(x)$.

(2) $d_1(x)$ and $d'_1(x)$ have a G.C.D. of degree $n_2=\sum_{k=2}^{n}(k-2)s_k$, say $d_2(x)$.

. . . .

(i) $d_{i-1}(x)$ and $d'_{i-1}(x)$ have a G.C.D. of degree

$$n_i = \sum_{k=i}^{n} (k-i)s_k, \text{ say } d_i(x).$$

These conditions are necessary because a root of multiplicity m in $g(x)=0$ is of multiplicity $m-1$ in $g'(x)=0$; and conversely, a root common to $g(x)=0$ and $g'(x)=0$ of multiplicity $m-1$ in $g'(x)=0$ is of multiplicity m in $g(x)=0$.

We shall now establish our main theorem: *If among all the possible choices for s_k, $i<k\leqq n$, there is only one choice giving rise to the number*

$$\sum_{k=i}^{n} (k-i)s_k, \quad 1\leqq i<n$$

and i is the least value for which this is true, then the conditions involved in (1), (2), $\cdots$, (i) *are necessary, sufficient, independent conditions that $f(x)=0$ possess s_k k-fold roots.* That there is a least value of $i\leqq n-1$ satisfying the stated condition appears from the fact that, for $i=n-1$, we have

$$n_i = \sum_{k=n-1}^{n} (k-n+1)s_k = s_n,$$

making only one choice of s_n possible. The sufficiency follows as soon as it is shown that all roots of $d_i(x)=0$ are common to $f(x)=0$, $f'(x)=0, \cdots, f^{(i)}(x)=0$, and conversely. This is true for $i=1$. Assuming it true for $i=r$, we have $f^{(r)}(x)=d_r(x)Q(x)$, whence

(A) $$f^{(r+1)}(x)=d_r(x)Q'(x)+d'_r(x)Q(x).$$

If $d_{r+1}(\xi)=0$, then, from the definition of d_{r+1}, $d_r(\xi)=d'_r(\xi)=0$, so that $f(\xi)=f'(\xi)=f''(\xi)=\cdots=f^{(r)}(\xi)=0$ because of the assumption and $f^{(r+1)}(\xi)=0$ because of (A). Conversely, if $f(\xi)=f'(\xi)=\cdots=f^{(r+1)}(\xi)=0$, then $d_r(\xi)=0$, so that by (A), $d'_r(\xi)Q(\xi)=0$. But $Q(\xi)\neq 0$ because $x-\xi$ cannot occur as a factor of $f^{(r)}(x)$ any more times than it occurs in $d_r(x)$. Hence $d'_r(\xi)=0$ and, therefore, $d_{r+1}(\xi)=0$. Since only one choice of the s_k can give rise to (i), this condition implies that $f^{(i)}(x)=0$ possess s_{i+r} roots of $f(x)=0$ to a multiplicity r, $r=1, 2, \cdots, n-i$. Hence $f(x)=0$ possesses these s_{i+r} roots to a multiplicity $r+i$. The condition (i) together with the condition ($i-1$) implies then that s_i roots of $f(x)=0$ are of multiplicity 1 in $f^{(i-1)}(x)=0$, and hence of multiplicity i in $f(x)=0$. Reasoning thus, by induction, we see that the i conditions set forth above imply that $f(x)=0$ possess s_k k-fold roots.

The independence of the conditions follows from the fact that condition (i) is obviously necessary to establish sufficiency, and i is the least value for which this is so. To impose these conditions in any

given case we make use of Article 69, Bôcher's *Introduction to Higher Algebra*. We give two examples.

Example 1. What are the necessary and sufficient conditions that

$$f(x) \equiv \sum_{i=0}^{n} a_i x^i = 0, \quad a_n \neq 0,$$

possess a root of multiplicity n ?

Here $\quad s_1 = s_2 = \cdots = s_{n-1} = 0; \quad s_n = 1; \quad n_1 = \sum_{k=1}^{n} (k-1)s_k = n-1.$

This is only true for this choice of the $s_k (k>1)$, and therefore we need only impose the conditions that $f(x)=0$ and $f'(x)=0$ have a G.C.D. of degree $(n-1)$. Following Article 69 in Bôcher, this implies the vanishing of $R=R_0, R_1, R_2, \cdots, R_{n-2}$, and the non-vanishing of R_{n-1}, a totality of n conditions, where R is the resultant of $f(x)$ and $f'(x)$ R_i is the ith sub-resultant. The condition $R_{n-1} \neq 0$ is equivalent to the condition $a_n \neq 0$, and the G.C.D. in this case is $f'(x)$ itself.

Example 2. What are the necessary and sufficient conditions for

$$f(x) \equiv \sum_{i=0}^{4} a_i x^i = 0, \quad a_4 \neq 0,$$

to possess one triple root and one single root?

Here $s_1 = 1,\ s_2 = 0,\ s_3 = 1,\ s_4 = 0;\ f(x) \equiv a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, $f'(x) \equiv 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$. The G.C.D. $d_1(x)$ of $f(x)$ and $f'(x)$ must have degree

$$n_1 = \sum_{k=1}^{4} (k-1)s_k = s_2 + 2s_3 + 3s_4 = 0.$$

Since the choice $s_2 = 2,\ s_3 = s_4 = 0$ could also satisfy this condition, i must be greater than 1. The G.C.D. of $d_1(x)$ and $d'_1(x)$ has degree

$$n_2 = \sum_{k=2}^{4} (k-2)s_k = s_3 + 2s_4 = 1.$$

Obviously $s_3 = 1,\ s_4 = 0$ is the only possible choice to satisfy this condition. Hence $i = 2$. To impose these conditions, we again refer to Bôcher, Article 69. The resultant of $f(x)$ and $f'(x)$ is the determinant R of the seventh order given by

$$R = \begin{vmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & 4a_4 & 3a_3 & 2a_1 & a_1 \\ 0 & 0 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 \\ 0 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 \\ 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 \end{vmatrix}, \quad R_1 = \begin{vmatrix} a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & 4a_4 & 3a_3 & 2a_1 \\ 0 & 4a_4 & 3a_3 & 2a_2 & a_1 \\ 4a_4 & 3a_3 & 2a_2 & a_1 & 0 \end{vmatrix}, \quad R_2 = \begin{vmatrix} a_4 & a_3 & a_2 \\ 0 & 4a_4 & 3a_3 \\ 4a_4 & 3a_3 & 2a_2 \end{vmatrix}.$$

We must have the three conditions $R=0$, $R_1=0$ $R_2\neq 0$, where R_1 and R_2 are the determinants indicated. $d_1(x)$ is then given by the determinant

$$d_1(x)=\begin{vmatrix} a_4 & a_3 & f(x) \\ 0 & 4a_4 & f'(x) \\ 4a_4 & 3a_3 & xf'(x) \end{vmatrix}$$

$$=a_4[(3a_3^2-8a_2a_4)x^2+(2a_2a_3-12a_1a_4)x+(a_1a_3-16a_0a_4)],$$

$$d'_1(x)=a_4[(6a_3^2-16a_2a_4)x+(2a_2a_3-12a_1a_4)].$$

The resultant R' of $d_1(x)$ and $d'_1(x)$ is

$$R'=(a_4)^3\begin{vmatrix} 3a_3^2-8a_2a_4 & 2a_2a_3-12a_1a_4 & a_1a_3-16a_0a_4 \\ 0 & 6a_3^2-16a_2a_4 & 2a_2a_3-12a_1a_4 \\ 6a_3^2-16a_2a_4 & 2a_2a_3-12a_1a_4 & 0 \end{vmatrix}$$

$R^1=0$ is the fourth and last condition we need impose, since $6a_3^2-16a_2a_4\neq 0$ because of the previous condition $R_2\neq 0$. We would, of course, have dropped the factor a_4 in $d_1(x)$.

Remark. Since the necessary and sufficient condition that two equations $f_1(x)=0$ and $f_2(x)=0$ possess a common root is so readily obtained, we might naturally think the same thing might be true in obtaining the conditions for three equations to possess a common root. However, the necessary and sufficient conditions in the latter case are obviously the same as the necessary and sufficient conditions that the G.C.D. of $f_1(x)$ and $f_2(x)$ equated to zero have a root in common with $f_3(x)=0$. To find the G.C.D. of $f_1(x)$ and $f_2(x)$ requires more specific information than that $f_1(x)=0$ and $f_2(x)=0$ have a root in common. In dealing with just two equations we do not need more specific information because we only need to insure that the G.C.D. of $f_1(x)$ and $f_2(x)$ be at least of the first degree—a much smaller requirement than that of the actual G.C.D. of $f_1(x)$ and $f_2(x)$.

No. 516. Proposed by *V. Thébault*, San Sebastian, Spain.

The lines joining a given point L to the vertices A, B, C, D of a tetrahedron $(T)=ABCD$ meet the circumsphere of (T) again in the points A', B', C', D'. Let M, M' be the isogonal conjugates of L for the tetrahedrons (T), $(T')=A'B'C'D'$; let P, Q, R, S and P', Q', R', S' be the projections of M and M' upon the faces of (T) and (T'). Show that the two tetrahedrons $PQRS$ and $P'Q'R'S'$ are similar.

Solution by the *Proposer*.

The faces of the pedal tetrahedron $PQRS$ of M for (T) are respectively perpendicular to the lines LA, LB, LC, LD (see, for instance,

Court's *Modern Pure Solid Geometry*, p. 242, Art. 742. Macmillan, 1935). Hence $PQRS$ is homothetic to the antipedal tetrahedron of L for (T), i. e., to the tetrahedron (U) formed by the planes perpendicular to the lines LA, LB, LC, LD at the vertices of (T).

Similarly the pedal tetrahedron $P'Q'R'S'$ of M' for (T') is homothetic to the antipedal tetrahedron (U') of L for (T'). Now the corresponding faces of the two tetrahedrons (U), (U') are obviously parallel, hence the propositon.

EDITORIAL NOTE. The two given tetrahedrons (T) and (T') are perspective from the center L. It follows from the above proof that the proposition is valid for any two perspective tetrahedrons and it is not necessary to restrict it to the case when the two tetrahedrons have a common circumsphere.

Moreover, the two tetrahedrons $PQRS$ and (U') are both homothetic to (U), whence $PQRS$ and (U') are homothetic. But $P'Q'R'S'$ is also homothetic to (U'), hence $PQRS$ and $P'Q'R'S'$ are homothetic. We may thus state the proposition: If two tetrahedrons (T) and (T'), are perspective, the projections upon their faces of the respective isogonal conjugates, for (T) and (T'), of their center of perspectivity form two tetrahedrons which are homothetic.—N. A. C.

No. 520. Proposed by *N. A. Court*, University of Oklahoma.

Two circles (A), (B) intersect in points E, F. Two lines passing through E intersect (A), (B) in points which determine the chords c,d. The parallels to c,d through E meet (B), (A) in the points Q,P, respectively. Show that the four points P,Q,F,cd are collinear.

Solution by *Walter B. Clarke*, San Jose, Calif.

Let AEB, $A'EB'$ be the given lines and let $AA'=c$, $BB'=d$ meet in $G=cd$. We have

$$\text{angle } EFA'=EAA', \quad \text{angle } EFB=GBE,$$

hence the angles $A'EB'$, AGB' are supplementary, and therefore the quadrilateral $A'FB'G$ is cyclic.

Let the parallel to $BB'G$ through E meet the line FG in P'. We have

$$\text{angle } EP'F=B'GF=B'A'F=EA'F.$$

Hence the four points E, F, P', A' are concyclic, and therefore the point P' coincides with P.

In a similar way it may be shown that the parallel to $AA'G$ through E meets the line FG in a point which coincides with Q. Hence the proposition.

Note. The figure may vary so as to require some slight modification of the above proof, but the method will remain the same.

Also solved by *Gerald B. Huff* who proved that the quadrilateral $A'FB'G$ is cyclic by applying Miquel's theorem to the triangle GAB and the points A', E, B' marked on its sides.

No. 548. Proposed by *N. A. Court*, University of Oklahoma.

Find the locus of the center of a variable sphere of fixed size which moves so that its radical plane with a fixed sphere passes through a fixed straight line.

I. Solution by *P. D. Thomas*, U. S. Navy.

Let the variable sphere have center M and fixed radius R; O is the center of the fixed sphere of fixed radius r. k is the fixed line.

If the radical plane is to pass through the fixed line k, then M must lie in a plane through O perpendicular to k, since the radical plane of two spheres is perpendicular to the line of centers.

The plane through O perpendicular to k meets k in the point P, and PO is a fixed distance.

The powers of P with respect to the variable and fixed spheres are respectively $PM^2=R^2$ and PO^2-r^2 which are equal since P lies always in the radical plane. Hence.

$$PM^2=R^2+OP^2-r^2.$$

Thus the locus of M is a circle with center P and radius squared equal to $R^2+OP^2-r^2$.

II. Solution by *Hyman J. Zimmerberg*, University of Chicago.

Without any loss of generality we can consider the fixed sphere as having its center at the origin and we can then denote its equation by $x^2+y^2+z^2-r^2=0$. Suppose that the fixed line has equations $x=x_0+at$, $y=y_0+bt$, $z=z_0+ct$, where (x_0, y_0, z_0) denotes a point on the line and a, b, c, is a set of direction numbers for the line. Let the equation of the variable sphere of fixed size be denoted by $(x-h)^2+(y-k)^2+(z-m)^2-R^2$.

The equation of the radical plane is then

$$2hx+2ky+2mz+R^2-h^2-k^2-m^2-r^2=0.$$

Substituting the equations of the fixed line into the above equation and demanding that the resulting equation be satisfied identically in t, we have

$$ha+kb+mc=0, \tag{1}$$

(2) $\qquad (h-x_0)^2+(k-y_0)^2+(m-z_0)^2=R^2-r^2+(x_0^2+y_0^2+z^2).$

Thus the required locus lies on the circle of intersection of the plane (1) with the sphere (2), wherein we consider h,k and m as running coordinates.

The plane (1) always exists as it is a plane through the origin perpendicular to the fixed line. Moreover by choosing the point (x_0, y_0, z_0) on the fixed line sufficiently far from the origin, the sphere (2) will also be real. The existence of the locus and the possibility of its being degenerate naturally depend upon the intersections of (1) and (2).

III. Solution by *Howard Eves*, Syracuse University.

Let the equations of the fixed and variable spheres be respectively

$$x^2+y^2+z^2=R^2,$$

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2,$$

and let the fixed line have equations

(1) $$x=0, \quad y=k.$$

The equation of the radical plane of the two spheres is

$$-2x_1x-2y_1y-2z_1z+(x_1^2+y_1^2+z_1^2-r^2+R^2)=0.$$

Since this plane is to pass through the line (1) the matrix

$$\begin{vmatrix} -2x_1 & -2y_1 & -2z_1 & x_1^2+y_1^2+z_1^2-r^2+R^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -k \end{vmatrix}$$

must be of rank 2. This implies that

$$z_1=0 \text{ and } x_1^2+(y_1-k)^2+z_1^2=r^2-R^2+k^2.$$

Hence the required locus is a circle in the xy-plane having its center at $(0, k, 0)$ and radius equal to $\sqrt{r^2-R^2+k^2}$.

Note. The corresponding theorem in the plane may be similarly established.

Also solved by *Earl V. Greer* and *J. S. Guérin*.

No. 567. Proposed by *Frank Hawthorne*, Allegheny College.

An alley has two poles placed against bottoms of buildings on each side and resting against the opposite building, the plane of the figures being perpendicular to the road and to both walls. The length

of the poles and the height of the point of intersection being given, obtain the equation which is satisfied by the width of the alley.

Solution by *Arthur Bernhart*, University of Oklahoma.

On line AHB erect perpendiculars AR, HQ, BS, with AS intersecting BR at Q. Let $a=AR$, $b=BS$, $h=HQ$, $r=RB$, $s=SA$, $x=HB$, $y=AH$, $w=AB$. Given h, r and s, it is required to find w.

From $w=x+y$ division by w yields

$$1=x/w+y/w=h/a+h/b,$$

in which fractions have been reduced to a common numerator by applying well-known proportions for similar triangles. This may be put in the form

$$1/h=1/a+1/b. \tag{1}$$

Thus h is the harmonic sum of a and b, where

$$a=\sqrt{r^2-w^2}, \quad b=\sqrt{s^2-w^2}.$$

The equation

$$1/h=1/\sqrt{r^2-w^2}+1/\sqrt{s^2-w^2} \tag{2}$$

can be rationalized to an equation of eighth degree in w (fourth degree in w^2). The desired quantity w is the positive root.

The extraneous roots introduced by rationalization may be avoided by a trigonometric substitution, incidentally exploiting the positive character of a, b, and h.

Suppose $r<s$, whence $a<b$, and set

$$s^2=r^2+t^2,$$

$$a=\sqrt{r^2-w^2}=t\tan\theta,$$

$$b=\sqrt{s^2-w^2}=t\sec\theta.$$

Equation (2) becomes $1/h=1/[t\tan\theta]+1/[t\sec\theta]$, or

$$\cot\theta+\cos\theta=t/h=\sqrt{s^2-r^2}/h. \tag{3}$$

Since the right hand member is a given positive number inspection of a table of natural trigonometric functions will indicate the value of θ. Since $\cot\theta$ and $\cos\theta$ must be positive we use acute angles only, and since $\cot\theta+\cos\theta$ is monotone decreasing from $+\infty$ to 0 as θ ranges from 0° to 90° there will be a unique root θ.

Since $$w^2=r^2-t^2\tan^2\theta=s^2-t^2\sec^2\theta,$$

$$w=+\sqrt{r^2-t^2\tan^2\theta}$$

uniquely determines w if the radicand is positive. Hence a real solution exists providing

$$t \tan \theta < r, \qquad t \sec \theta < s,$$

$$1/[t \tan \theta] > 1/r, \qquad 1/[t \sec \theta] > 1/s.$$

Therefore $$1/h = 1/[t \tan \theta + 1/[t \sec \theta] > 1/r + /1s,$$

showing that h must be given less than the harmonic sum of r and s. If $r = s$, then $a = b = 2h$, and the Pythagorean theorem provides w.

Note that equation (1) is valid for any trapezoid, without requiring AB to be perpendicular to its bases. It provides the physicist the formula for resistances in parallel, and for condensers in series. For the air pilot h is the radius of action (per unit of time) when a and b are the ground speeds out and return. Conversely the trapezoid construction provides a simple graphical solution for these problems.

Also solved by *Ferrel Atkins*, *M. I. Chernofsky*, *Howard Eves*, *H. E. Fettis*, *C. S. Larkey*, *W. S. Loud*, *A. Sisk*, *P. D. Thomas*, *H. J. Zimmerberg*, and one reader whose name is missing from the page. Most solvers gave the rational form of equation (2) above. Fettis introduced the variable $z = ab$ and reduced (2) to the form

$$z^4 - 4h^2z^2 = h^2(s^2 - r^2)^2.$$

After z is determined, w may be found as a root of $(r^2 - w^2)(s^2 - w^2) = z^2$, a quadratic in w^2.

Another aspect of this "alley problem" is given in the *American Mathematical Monthly*, April, 1941, p. 268.

PROPOSALS

No. 585. Proposed by *John H. White*, Summit, New Jersey.

Two i lands, A, B, (considered as points) in a circular lake (O) are equ distant from the center O. Determine the point P on the shore such that the trip $AP + PB$ shall be a minimum. Obtain a geometric construction for the point P.

No. 586. Proposed by *N. A. Court*, University of Oklahoma.

If the edges of a tetrahedron are coplanar with the polar lines for the circumsphere of the tetrahedron, of the respective edges, then the three products of the three pairs of opposite edges of the tetrahedron are equal.

No. 687. Proposed by *P. D. Thomas*, U. S. Navy.

Find the equation of the ruled surface generated by a variable line which meets both the Z-axis and the curve $x=a\sin u$, $y=b\cos u$, $z=c\sin u\cos u$, and remains parallel to the XY-plane.

No. 588. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Find a perfect square of twelve digits formed from the juxtaposition of two squares, one having four digits and the other eight.

No. 689. Proposed by *H. S. Grant*, Rutgers University.

$$n-\frac{n(n^2-1^2)}{3!}+\frac{n(n^2-1^2)(n^2-3^2)}{5!}-\cdots=\pm 1 \text{ or } 0,$$

according as n is odd or even.

No. 590. Proposed by *Nev. R. Mind*.

The midpoint of the segment joining the orthocenter of a triangle to the trace on the circumcircle of a bisector of an angle of the triangle has equal powers with respect to the two tritangent circles (i. e. circles touching the three sides of the triangle having their centers on the bisector considered.